Response of an Elastic Half Space to an Arbitrary 3-D Vector Body Force

We wish to calculate the displacement vector **u***(x,y,z)* on the surface of the Earth due to a vector body force at depth. The approach will be used to simulate both volcanic inflation events with arbitrary source shape and motion on curved and discontinuous faults. For simplicity, we ignore the effects of sphericity. We assume a Poisson material, and maintain constant moduli with depth. The method follows:

i.) Develop the three differential equations relating 3-D vector displacement to a 3-D vector body force.
ii.) Take the 3-D Fourier transform to reduce the partial differential equation to a set of linear algebraic equations.
iii.) Solve the linear system using the symbolic capabilities in Matlab.
iv.) Perform the inverse fourier transform in the *z*-direction (depth) by repeated application of the Cauchy Residue Theorem.
v.) Check the analytic solution using the symbolic capabilities in Matlab.
vi.) Solve the Boussinesq Problem to correct the non-zero normal traction on the half-space
vii.) Construct screw dislocation and test with analytic line-source solution.
viii.) Integrate the point-source Green's function to simulate a vertical fault and check with the analytic fault-plane solution.
ix.) 2-D analytic test from 3-D solution
x.) Develop an equivalent body force for a general fault model.
xi.) Force couples on a regular grid
xii.) Fortran source code for elastic half-space model
xiii.) Modify the solution to have a layered half-space (see viscoelastic derivation)
xiv.) Modify the solution for viscoelastic rheology (see viscoelastic derivation)
xv.) Modify the solution to account for surface topography

i) ***Develop differential equations***

The following equations relate stress to body forces, stress to strain, and strain to displacement:

$$\tau_{ij,j} = -\rho_i \tag{1}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2}$$

$$\tau_{ij} = \delta_{ij}\lambda\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

substitute (2) into (3): (3)

$$\tau_{ij} = \delta_{ij}\lambda u_{k,k} + \mu\left(u_{i,j} + u_{j,i}\right) \tag{4}$$

substitute (4) into (1):

$$\delta_{ij}\lambda u_{k,kj} + \mu\left(u_{i,jj} + u_{j,ij}\right) = -\rho_i \tag{5}$$

or

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} = -\rho_i \tag{6}$$

Expand (6) into three equations for *u, v, w*:

$$\mu\nabla^2 u + (\lambda + \mu)\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y dx} + \frac{\partial^2 w}{\partial z \partial x}\right] = -\rho_x$$

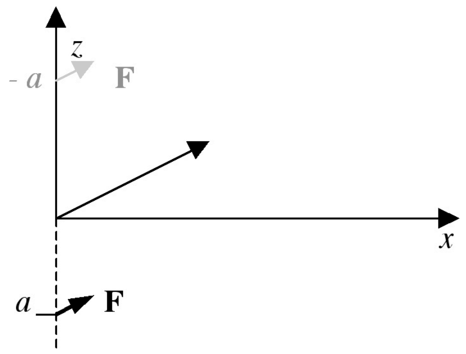
$$\mu\nabla^2 v + (\lambda + \mu)\left[\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z \partial y}\right] = -\rho_y \tag{7}$$

$$\mu\nabla^2 w + (\lambda + \mu)\left[\frac{\partial^2 u}{\partial x dz} + \frac{\partial^2 v}{\partial y dz} + \frac{\partial^2 w}{\partial z^2}\right] = -\rho_z$$

ii) ***3-D Fourier transform***

We assume a point vector body force applied at $x = 0$, $y = 0$, and $z = a$. In order to simulate a half-space, we also include the effects of an image source at $x = 0$, $y = 0$, and $z = -a$ [*Weertman*, 1964]. Equation 8 describes a point body force at both source and image locations, where **F** is a vector force with units of force.

$$\rho(x,y,z) = \mathbf{F}\delta(x)\delta(y)\delta(z-a) + \mathbf{F}\delta(x)\delta(y)\delta(z+a) \tag{8}$$



We then take the 3-D Fourier transform of equation 7 in order to simplify the set of differential equations into a algebraic equations. The following identities are used:

$$\Im\left[\frac{\partial u(\mathbf{x})}{\partial x}\right] = i2\pi k_x U(\mathbf{k})$$

$$\Im[\delta(x)] = 1 \tag{9}$$

$$\Im[\delta(z-a) + \delta(z+a)] = e^{-2\pi i k_z a} + e^{2\pi i k_z a}$$

After Fourier transforming, the linear system of equations then becomes

$$(\lambda+\mu)\begin{bmatrix} k_x^2 + \dfrac{\mu \check{\mathbf{k}}^2}{(\lambda+\mu)} & k_y k_x & k_z k_x \\ k_x k_y & k_y^2 + \dfrac{\mu \check{\mathbf{k}}^2}{(\lambda+\mu)} & k_z k_y \\ k_x k_z & k_y k_z & k_z^2 + \dfrac{\mu \check{\mathbf{k}}^2}{(\lambda+\mu)} \end{bmatrix} \begin{bmatrix} U(\mathbf{k}) \\ V(\mathbf{k}) \\ W(\mathbf{k}) \end{bmatrix} = \frac{e^{-i2\pi k_z a} + e^{+i2\pi k_z a}}{4\pi^2} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \tag{10}$$

where $\check{\mathbf{k}} = (k_x, k_y, k_z)$ and $\check{\mathbf{k}}^2 = \check{\mathbf{k}} \bullet \check{\mathbf{k}}$.

iii) ***Solve the linear system using Matlab***

The displacement solution to equation 10 is found using the symbolic library in Matlab or Mathematica:

$$\begin{bmatrix} U(\mathbf{k}) \\ V(\mathbf{k}) \\ W(\mathbf{k}) \end{bmatrix} = \frac{(\lambda+\mu)}{\check{\mathbf{k}}^4\mu(\lambda+2\mu)} \begin{bmatrix} \left(k_y^2+k_z^2\right)+\dfrac{\mu\check{\mathbf{k}}^2}{(\lambda+\mu)} & -k_yk_x & -k_zk_x \\ -k_xk_y & \left(k_x^2+k_z^2\right)+\dfrac{\mu\check{\mathbf{k}}^2}{(\lambda+\mu)} & -k_zk_y \\ -k_xk_z & -k_yk_z & \left(k_x^2+k_y^2\right)+\dfrac{\mu\check{\mathbf{k}}^2}{(\lambda+\mu)} \end{bmatrix} \frac{(e^{-i2\pi k_z a}+e^{i2\pi k_z a})}{4\pi^2} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \tag{11}$$

The following Matlab symbolic code verifies that this inversion is correct.

```
%  Matlab routine to check the solution in the kx, ky, kz domain
%
%  wavenumbers
    pi = sym('pi');
    kx=sym('kx');
    ky=sym('ky');
    kz=sym('kz');
%  elastic constants
    la=sym('la');
    mu=sym('mu');
    lam=la+mu;
%  combinations of wavenumbers
    c=sym('c');
    c=(kx*kx+ky*ky+kz*kz);
%  forward matrix
    A=[kx*kx+mu*c/lam, ky*kx,               kz*kx;
       kx*ky,          ky*ky+mu*c/lam,      kz*ky;
       kx*kz,          ky*kz,        kz*kz+mu*c/lam];
%  solution in Fourier domain, inverse matrix
    B=[c*mu/lam+ky*ky+kz*kz,  -kx*ky,              -kx*kz;
       -kx*ky,         c*mu/lam+kx*kx+kz*kz,        -kz*ky;
       -kx*kz,                -ky*kz,  c*mu/lam+kx*kx+ky*ky];
%  normalize
    A=A*lam;
    B=B*lam/(mu*(la+2*mu)*c*c);
%  multiply to get the identity matrix
    C=B*A;
    simplify(C)
ans  =  [ 1,  0,  0]
        [ 0,  1,  0]
        [ 0,  0,  1]
```

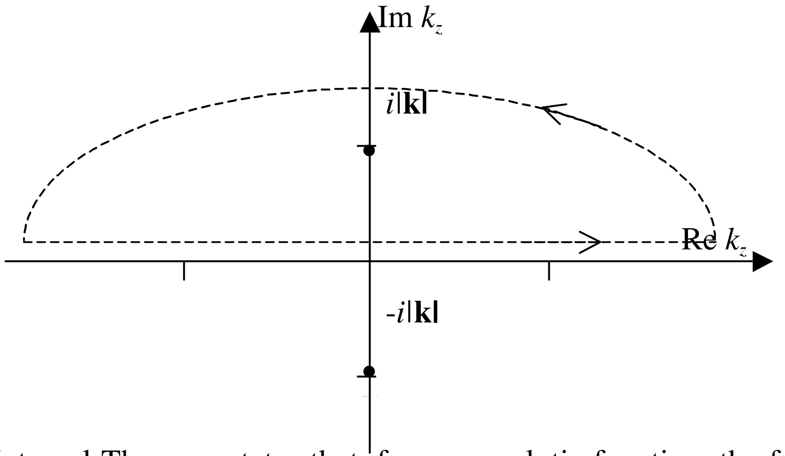
iv) ***Perform the inverse Fourier transform along the k_z axis***

We apply the Cauchy Residue Theorem to perform the inverse Fourier transform of each term, for example $V_x(\mathbf{k})$. Our method involves finding the poles of V_x, and using calculus of residues to integrate with respect to k_z. The first step is to separate the denominator into its poles:

$$V(\mathbf{k},z) = -F_x \frac{k_x k_y}{4\pi^2} \frac{(\lambda+\mu)}{\mu(\lambda+2\mu)} \int_{-\infty}^{\infty} \frac{e^{i2\pi k_z(z-a)}}{\left(k_x^2+k_y^2+k_z^2\right)^2} dk_z$$
$$= -F_x \frac{k_x k_y}{4\pi^2} \frac{(\lambda+\mu)}{\mu(\lambda+2\mu)} \int_{-\infty}^{\infty} \frac{e^{i2\pi k_z(z-a)}}{\left(k_z+i|\mathbf{k}|\right)^2\left(k_z-i|\mathbf{k}|\right)^2} dk_z \tag{12}$$

where $\mathbf{k} = (k_x, k_y)$ and $|\mathbf{k}| = \left(k_x^2+k_y^2\right)^{1/2}$.

Here the denominator has four poles total, two at $k_z = -i|\mathbf{k}|$, and two at $k_z = i|\mathbf{k}|$. We will consider two cases $z > a$, and $z < a$. First consider the case for $z > a$. We have factored the denominator, and have recognized that the integrand will vanish for large, positive z if we close the contour in the upper hemisphere:



The Cauchy Integral Theorem states that, for any analytic function, the following holds for a counter-clockwise path surrounding the pole:

$$\oint \frac{f(z)}{z-z_o} dz = i2\pi \, \text{Residue}\left[f(z), z_o\right] \tag{13}$$

In the special case where there are poles of order greater than one, we compute the residue by the following:

(14)

$$\text{Res}\left[f(z),z_o\right] = \frac{1}{m-1}\lim_{z\to z_o}\frac{d^q}{dz^q}\left[\left(z-z_o\right)^m f(z)\right]$$

where z_o = pole = $\pm i|\mathbf{k}|$

m = order = 2

$q = m-1$ (the q th derivative)

To satisfy the boundary condition of $z > a$ (Im $k_z > 0$), we must integrate around the positive pole $(k_z = i|\mathbf{k}|)$.

$$V(\mathbf{k},z) = -F_x\frac{k_x k_y}{4\pi^2}\frac{(\lambda+\mu)}{\mu(\lambda+2\mu)}2\pi i\ \text{Res}\left[f(k_z),i|\mathbf{k}|\right]$$

$$\text{where } f(k_z) = \frac{e^{i2\pi k_z(z-a)}}{(k_z+i|\mathbf{k}|)^2(k_z-i|\mathbf{k}|)^2}$$

$$\text{Re}\,s\left[f(k_z),i|\mathbf{k}|\right] = \frac{1}{2-1}\lim_{k_z\to i\mathbf{k}}\frac{d^{2-1}}{dk_z^{2-1}}\left[\frac{(k_z-i|\mathbf{k}|)^2 e^{i2\pi k_z(z-a)}}{(k_z+i|\mathbf{k}|)^2(k_z-i|\mathbf{k}|)^2}\right]$$

$$= \frac{-2e^{-2\pi|\mathbf{k}|(z-a)}+2\pi i(z-a)e^{-2\pi|\mathbf{k}|(z-a)}}{(2i|\mathbf{k}|)^3}$$

$$V(\mathbf{k},z) = -F_x\frac{k_x k_y}{2\pi}\frac{(\lambda+\mu)}{\mu(\lambda+2\mu)}\left(\frac{1+2\pi|\mathbf{k}|(z-a)}{4|\mathbf{k}|^3}\right)e^{-2\pi|\mathbf{k}|(z-a)} \qquad (15)$$

Next consider $z < a$. For this case, we must close the integration path in the lower hemisphere to satisfy the boundary conditions, whereby the only contribution from this integration will be from the $-i|\mathbf{k}|$ pole. This result will be

$$V(\mathbf{k},z) = -F_x\frac{k_x k_y}{2\pi}\frac{(\lambda+\mu)}{\mu(\lambda+2\mu)}\left(\frac{1+2\pi|\mathbf{k}|(-z-a)}{4|\mathbf{k}|^3}\right)e^{-2\pi|\mathbf{k}|(-z-a)}. \qquad (16)$$

After completing the integration over the poles in the complex plane for all components of equation 11, the body-force solutions for the half-space containing the source are

$$\begin{bmatrix} U(\mathbf{k},z) \\ V(\mathbf{k},z) \\ W(\mathbf{k},z) \end{bmatrix}^{source} = B_s C \begin{bmatrix} D + \frac{k_y^2}{|\mathbf{k}|^2} - 2\pi|\mathbf{k}|(z-a)\frac{k_x^2}{|\mathbf{k}|^2} & \frac{-k_x k_y}{|\mathbf{k}|^2}\{1 + 2\pi|\mathbf{k}|(z-a)\} & -i2\pi k_x(z-a) \\ \frac{-k_x k_y}{|\mathbf{k}|^2}\{1 + 2\pi|\mathbf{k}|(z-a)\} & D + \frac{k_x^2}{|\mathbf{k}|^2} - 2\pi|\mathbf{k}|(z-a)\frac{k_y^2}{|\mathbf{k}|^2} & -i2\pi k_y(z-a) \\ -i2\pi k_x(z-a) & -i2\pi k_y(z-a) & D + 2\pi|\mathbf{k}|(z-a) \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}.$$

The solutions for the half-space containing the image are

$$\begin{bmatrix} U(\mathbf{k},z) \\ V(\mathbf{k},z) \\ W(\mathbf{k},z) \end{bmatrix}^{image} = B_i C \begin{bmatrix} D + \frac{k_y^2}{|\mathbf{k}|^2} - 2\pi|\mathbf{k}|(-z-a)\frac{k_x^2}{|\mathbf{k}|^2} & \frac{-k_x k_y}{|\mathbf{k}|^2}\{1 + 2\pi|\mathbf{k}|(-z-a)\} & -i[2\pi k_x(-z-a)] \\ \frac{-k_x k_y}{|\mathbf{k}|^2}\{1 + 2\pi|\mathbf{k}|(-z-a)\} & D + \frac{k_x^2}{|\mathbf{k}|^2} - 2\pi|\mathbf{k}|(-z-a)\frac{k_y^2}{|\mathbf{k}|^2} & -[i2\pi k_y(-z-a)] \\ i2\pi k_x(-z-a) & i2\pi k_y(-z-a) & -[D + 2\pi|\mathbf{k}|(-z-a)] \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

$$\text{where} \quad B_s = \frac{e^{-2\pi|\mathbf{k}|(z-a)}}{2\pi|\mathbf{k}|}, \quad B_i = \frac{e^{-2\pi|\mathbf{k}|(-z-a)}}{2\pi|\mathbf{k}|}, \quad C = \frac{(\lambda+\mu)}{4\mu(\lambda+2\mu)}, \quad D = \frac{\lambda+3\mu}{\lambda+\mu}. \tag{17}$$

We emphasize the negative sign in front of the image F_z components, due to the nature of the image source vector, which has the opposite vertical component sign in the source space.

In addition to the displacements, we will also need the vertical derivative of the vertical displacement for the F_z body force only. The horizontal derivatives are simply $i2\pi\mathbf{k}$ times the displacement. The result is

$$\frac{\partial W}{\partial z} = -C\left\{e^{-2\pi|\mathbf{k}|(z-a)}\left[D + 2\pi|\mathbf{k}|(z-a) - 1\right] + e^{-2\pi|\mathbf{k}|(-z-a)}\left[D + 2\pi|\mathbf{k}|(-z-a) - 1\right]\right\}$$

We then add both body-force solutions for the source-space and the image-space to obtain our final source-image solution:

$$\begin{bmatrix} U(\check{\mathbf{k}},z) \\ V(\check{\mathbf{k}},z) \\ W(\check{\mathbf{k}},z) \end{bmatrix} = \begin{bmatrix} U(\check{\mathbf{k}},z) \\ V(\check{\mathbf{k}},z) \\ W(\check{\mathbf{k}},z) \end{bmatrix}^{source} + \begin{bmatrix} U(\check{\mathbf{k}},z) \\ V(\check{\mathbf{k}},z) \\ W(\check{\mathbf{k}},z) \end{bmatrix}^{image} \tag{18}$$

When we set $z=0$ then the solution is

$$\begin{bmatrix} U(\mathbf{k}) \\ V(\mathbf{k}) \\ W(\mathbf{k}) \end{bmatrix} = 2BC \begin{bmatrix} D + \dfrac{k_y^2}{|\mathbf{k}|^2} + 2\pi|\mathbf{k}|a\dfrac{k_x^2}{|\mathbf{k}|^2} & \dfrac{-k_x k_y}{|\mathbf{k}|^2}\left\{1 - 2\pi|\mathbf{k}|a\right\} & i2\pi k_x a \\ \dfrac{-k_x k_y}{|\mathbf{k}|^2}\left\{1 - 2\pi|\mathbf{k}|a\right\} & D + \dfrac{k_x^2}{|\mathbf{k}|^2} + 2\pi|\mathbf{k}|a\dfrac{k_y^2}{|\mathbf{k}|^2} & i2\pi k_y a \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \qquad (19)$$

where $B = \dfrac{e^{2\pi|\mathbf{k}|a}}{2\pi|\mathbf{k}|}$, $C = \dfrac{(\lambda + \mu)}{4\mu(\lambda + 2\mu)}$, $D = \dfrac{\lambda + 3\mu}{\lambda + \mu}$.

v) ***Check the analytic solution using the symbolic capabilities in Matlab***

Again we can use the symbolic capabilities in Matlab to check these answers. In this case we insert this solution into the original differential equation where the equations have been Fourier transformed in the *x*- and *y*- directions. The vertical derivatives are carried out symbolically!

```
%  matlab routine to check the body-force solution
%
    pi = sym('pi');
    pi2= 4*pi*pi;
%  wavenumbers and observation depth z
    kx=sym('kx');
    ky=sym('ky');
    kh2=kx*kx+ky*ky;
    kh=sqrt(kh2);
%  source depth and elastic constants
    a=sym('a');
    z =sym('z');
    z =a;
    la=sym('la');
    mu=sym('mu');
    lam=la+mu;
%  solution in Fourier domain after z-integration
    B=exp(-2*pi*kh*(z-a))/(2*pi*kh);
    C=lam/(4*mu*(la+2*mu));
    D=(la+3*mu)/lam;
%  x-body force
    Ux =  B*C*(D + (ky*ky)/kh2 - 2*pi*kh*(z-a)*kx*kx/kh2);
    Vx = -B*C*kx*ky*(1+2*pi*kh*(z-a))/kh2;
    Wx = -B*C*i*2*pi*kx*(z-a);
%  y-body force
    Uy = -B*C*kx*ky*(1+2*pi*kh*(z-a))/kh2;
    Vy =  B*C*(D + (kx*kx)/kh2 - 2*pi*kh*(z-a)*ky*ky/kh2);
    Wy = -B*C*i*2*pi*ky*(z-a);
%  z-body force
    Uz = -B*C*i*2*pi*kx*(z-a);
    Vz = -B*C*i*2*pi*ky*(z-a);
    Wz =  B*C*(D + 2*pi*kh*(z-a));
% add the soLutions
    U = Ux + Uy + Uz;
    V = Vx + Vy + Vz;
    W = Wx + Wy + Wz;
% compute rhx
    RHX = mu*(-pi2*kx*kx*U-pi2*ky*ky*U+diff(U,'z',2));
    RHX = RHX + lam*(-pi2*kx*kx*U-pi2*kx*ky*V+i*2*pi*kx*diff(W,'z',1));
simplify(RHX)
ans  = 0
% compute rhy
    RHY = mu*(-pi2*kx*kx*V-pi2*ky*ky*V+diff(V,'z',2));
    RHY = RHY + lam*(-pi2*kx*ky*U-pi2*ky*ky*V+i*2*pi*ky*diff(W,'z',1));
%
simplify(RHY)
ans  = 0
```

```
% compute rhz
```

```
    RHZ = mu*(-pi2*kx*kx*W-pi2*ky*ky*W+diff(W,'z',2));
    RHZ = RHZ + lam*(i*2*pi*kx*diff(U,'z',1)+i*2*pi*ky*diff(V,'z',1)+diff(W,'z',2));
simplify(RHZ)
ans  =  0
```

vi) *Determine the surface stress distribution on an elastic half-space to satisfy the boundary condition - The Boussinesq Problem*

We must satisfy the surface boundary condition, that being that the shear stress must be equal to zero. We have included an image source to *partially* satisfy this boundary condition, but by taking the derivative of the full solution (source and image) with respect to *z* and evaluating at *z=0* we find that while the shear tractions disappears, the normal tractions are actually doubled!

$$\begin{bmatrix} \tau_{xz}(\mathbf{k}) \\ \tau_{yz}(\mathbf{k}) \\ \tau_{zz}(\mathbf{k}) \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = e^{\beta a} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -i\frac{k_x}{|\mathbf{k}|}\left(\alpha\beta a + \frac{\mu}{(\lambda+2\mu)}\right) & -i\frac{k_y}{|\mathbf{k}|}\left(\alpha\beta a + \frac{\mu}{(\lambda+2\mu)}\right) & (\alpha\beta a - 1) \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

$$\text{where} \quad \alpha = \frac{\lambda+\mu}{\lambda+2\mu} \qquad \beta = 2\pi|\mathbf{k}| \tag{20}$$

The following Matlab code verifies this computation of the stress tensor:

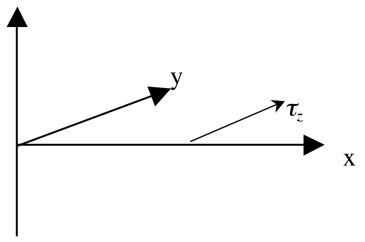
```
%  matlab routine to compute stress tensor (specifically stresses Txz Tyz Tzz) before and
after we incorporate the  image

%  this computation produces a doubled vertical stress component that requires a Boussinesq
correction
    pi = sym('pi');
    pi2= 4*pi*pi;
%  wavenumbers and observation depth z
    kx=sym('kx');
    ky=sym('ky');
    kh2=kx*kx+ky*ky;
    kh=sqrt(kh2);
%  source depth and elastic constants
    a=sym('a');
    z =sym('z');
    la=sym('la');
    mu=sym('mu');
    lam=la+mu;
%  solution in Fourier domain after z-integration
    Bs=exp(-2*pi*kh*(z-a))/(2*pi*kh);
    Bi=exp(-2*pi*kh*(-z-a))/(2*pi*kh);
    C=lam/(4*mu*(la+2*mu));
    D=(la+3*mu)/lam;
% x-component
    Uxs =  Bs*C*(D + (ky*ky)/kh2 - 2*pi*kh*(z-a)*kx*kx/kh2);
    Uxi =  Bi*C*(D + (ky*ky)/kh2 - 2*pi*kh*(-z-a)*kx*kx/kh2);
    Ux  =  Uxs + Uxi;
    Uys = -Bs*C*kx*ky*(1+2*pi*kh*(z-a))/kh2;
    Uyi = -Bi*C*kx*ky*(1+2*pi*kh*(-z-a))/kh2;
    Uy  =  Uys + Uyi;
    Uzs = -Bs*C*i*2*pi*kx*(z-a);
    Uzi = -(Bi*C*i*2*pi*kx*(-z-a));
    Uz  =  Uzs + Uzi;
% y-component
    Vx  =  Uy;
    Vys =  Bs*C*(D + (kx*kx)/kh2 - 2*pi*kh*(z-a)*ky*ky/kh2);
    Vyi =  Bi*C*(D + (kx*kx)/kh2 - 2*pi*kh*(-z-a)*ky*ky/kh2);
    Vy  =  Vys + Vyi;
    Vzs = -Bs*C*i*2*pi*ky*(z-a);
    Vzi = -(Bi*C*i*2*pi*ky*(-z-a));
    Vz  =  Vzs + Vzi;
% z-component
    Wxs = -Bs*C*i*2*pi*kx*(z-a);
    Wxi =  Bi*C*i*2*pi*kx*(-z-a);
    Wx  =  Wxs + Wxi;
    Wys = -Bs*C*i*2*pi*ky*(z-a);
    Wyi =  Bi*C*i*2*pi*ky*(-z-a);
    Wy  =  Wys + Wyi;
    Wzs =  Bs*C*(D + 2*pi*kh*(z-a));
    Wzi = -(Bi*C*(D + 2*pi*kh*(-z-a)));
    Wz  =  Wzs + Wzi;
%  source
%  now compute the derivative of the displacements with respect to z
    sUxz = diff(Uxs,'z');
    sUyz = diff(Uys,'z');
    sUzz = diff(Uzs,'z');
    sVxz = diff(Uys,'z');
    sVyz = diff(Vys,'z');
    sVzz = diff(Vzs,'z');
   sWxz = diff(Uzs,'z');
    sWyz = diff(Vzs,'z');
    sWzz = diff(Wzs,'z');
%  source + image
%  now compute the derivative of the displacements with respect to z
    Uxz = diff(Ux,'z');
    Uyz = diff(Uy,'z');
    Uzz = diff(Uz,'z');
    Vxz = diff(Vx,'z');
    Vyz = diff(Vy,'z');
    Vzz = diff(Vz,'z');
    Wxz = diff(Wx,'z');
    Wyz = diff(Wy,'z');
    Wzz = diff(Wz,'z');
%  source
%  now compute the stresses, tau-xz  tau-yz  tau-zz
  sT11=mu*(limit(sUxz+i*2*pi*kx*Uzs,'z',0));
  sT12=mu*(limit(sUyz+i*2*pi*kx*Vzs,'z',0));
  sT13=mu*(limit(sUzz+i*2*pi*kx*Wzs,'z',0));
  sT21=mu*(limit(sVxz+i*2*pi*ky*Uzs,'z',0));
  sT22=mu*(limit(sVyz+i*2*pi*ky*Vzs,'z',0));
  sT23=mu*(limit(sVzz+i*2*pi*ky*Wzs,'z',0));
  sT31=(la + 2*mu)*limit(sWxz,'z',0)+la*limit(i*2*pi*kx*Uxs+i*2*pi*ky*Uys,'z',0);
  sT32=(la + 2*mu)*limit(sWyz,'z',0)+la*limit(i*2*pi*kx*Uys+i*2*pi*ky*Vys,'z',0);
 sT33=(la + 2*mu)*limit(sWzz,'z',0)+la*limit(i*2*pi*kx*Uzs+i*2*pi*ky*Vzs,'z',0);
```

```
%  source + image
       T11=mu*(limit(Uxz+i*2*pi*kx*Wx,'z',0));
       T12=mu*(limit(Uyz+i*2*pi*kx*Wy,'z',0));
       T13=mu*(limit(Uzz+i*2*pi*kx*Wz,'z',0));
       T21=mu*(limit(Vxz+i*2*pi*ky*Wx,'z',0));
       T22=mu*(limit(Vyz+i*2*pi*ky*Wy,'z',0));
       T23=mu*(limit(Vzz+i*2*pi*ky*Wz,'z',0));
       T31=(la + 2*mu)*limit(Wxz,'z',0)+la*limit(i*2*pi*kx*Ux+i*2*pi*ky*Vx,'z',0);
       T32=(la + 2*mu)*limit(Wyz,'z',0)+la*limit(i*2*pi*kx*Uy+i*2*pi*ky*Vy,'z',0);
       T33=(la + 2*mu)*limit(Wzz,'z',0)+la*limit(i*2*pi*kx*Uz+i*2*pi*ky*Vz,'z',0);
%
%  compute stress tensor
%
%  source stress tensor:
       sourceTauz=simplify([sT11, sT12, sT13; sT21, sT22, sT23; sT31, sT32, sT33]);
%
%  source + image stress tensor:
       source_imageTauz=simplify([T11, T12, T13; T21, T22, T23; T31, T32, T33])
%
ans:   source_imageTauz    =     [0,  0,  0]
                                 [0,  0,  0]
[-i*kx*exp(2*a*pi*(kx^2+ky^2)^(1/2))*(mu*(kx^2+ky^2)^(1/2)+
2*pi*a*kx^2*mu+2*pi*a*ky^2*mu+2*la*pi*a*kx^2+2*la*pi*a*ky^2)/(kx^2+ky^2)/(la+2*mu)
,
-i*ky*exp(2*a*pi*(kx^2+ky^2)^(1/2))*(mu*(kx^2+ky^2)^(1/2)+
2*pi*a*kx^2*mu+2*pi*a*ky^2*mu+2*la*pi*a*kx^2+2*la*pi*a*ky^2)/(la+2*mu)/(kx^2+ky^2)
,                                       exp(2*a*pi*(kx^2+ky^2)^(1/2))*(-la-
2*mu+2*la*a*pi*(kx^2+ky^2)^(1/2)+2*mu*a*pi*(kx^2+ky^2)^(1/2))/(la+2*mu)]
%
%  compute ((source + image tensor) - 2*source tensor)
%  this should be zero to prove that stress tensor z componets are twice that of the source
   components with only source and image solutions.
%
doubleTau=simplify([T11 T12 T13; T21 T22 T23; (T31-2*sT31) (T32-2*sT32) (T33-2*sT33)])
%
ans:   doubleTau   =   [ 0,  0,  0]
                       [ 0,  0,  0]
                       [ 0,  0,  0]
```

We must impose a negative surface traction to match the boundary condition. We will solve the Boussinesq Problem in order to satisfy the surface boundary condition, for which an arbitrary traction is applied to an elastic half-space.

The following derivation follows the approach of *Steketee, J.A., On Volterra's dislocations in a semi-infinite elastic medium, Can. J. Phys., V 36, p. 192-205, 1958.*

From the above full space solution, including both source and image at a < 0, a > 0 respectively, we found that the zero-traction surface boundary condition is only *partially* satisfied. The remaining surface traction is $\tau_z(\mathbf{k})$.



We must impose a negative surface traction to match the boundary condition. To do this we solve the Boussinesq problem where an arbitrary traction is applied to an elastic half-space. Our approach is to find the Galerkin vector $\boldsymbol{\Gamma}_i$, such that we may define complimentary solutions satisfying the boundary condition of $\tau_{i,3}(x,y,0) = -\tau_3$ for both displacements and stresses:

$$\begin{aligned} u_i &= \Gamma_{i,kk} - \alpha\Gamma_{k,ki} \\ \tau_{ij} &= \lambda(1-\alpha)\delta_{ij}\Gamma_{k,kll} + \mu\left(\Gamma_{i,jkk} + \Gamma_{j,ikk}\right) - 2\mu\alpha\Gamma_{k,kij} \\ &\text{where } \alpha = \frac{(\lambda+\mu)}{(\lambda+2\mu)}. \end{aligned} \tag{21}$$

We note that $\tau_{ij,j} = 0$ becomes the biharmonic equation for Γ, $\Gamma_{\text{i,jjkk}} = 0$, and we assume $\Gamma_1 = \Gamma_2 = 0$ so only Γ_3 will be used, which we will now call Γ.

Take the 2- D Fourier transform of the biharmonic equation:

$$\Gamma_{\text{i,jjkk}} = \frac{\partial}{\partial x^2}\left(\nabla^2\Gamma\right) + \frac{\partial}{\partial y^2}\left(\nabla^2\Gamma\right) + \frac{\partial}{\partial z^2}\left(\nabla^2\Gamma\right) = 0 \tag{22}$$

A general solution to this problem is:

$$\Gamma(\mathbf{k},\text{z}) = \left(A + B(2\pi|\mathbf{k}|z)\right)e^{2\pi|\mathbf{k}|z} + \left(C + D(2\pi|\mathbf{k}|z)\right)e^{-2\pi|\mathbf{k}|z} \tag{23}$$

Our solution will be valid for z < 0 so we need C = D = 0 in order to suppress the two solutions that diverge as z $\rightarrow -\infty$. We will use the traction $-\tau_3(\mathbf{k})$ to balance the stesses arising from the forces F_x, F_y, and F_z.

Boussinesq Solution $\tau_{xz}\big|_0 = \tau_{yz}\big|_0 = 0 \qquad \tau_{zz}\big|_0 = -\tau_3$

$$U_B = -\alpha\frac{\partial^2\Gamma}{\partial x\partial z} \qquad V_B = -\alpha\frac{\partial^2\Gamma}{\partial y\partial z} \qquad W_B = -\alpha\frac{\partial^2\Gamma}{\partial z^2} + \nabla^2\Gamma$$
$$\tau_{xz} = \mu\left(\frac{\partial U}{\partial z} + \frac{\partial U}{\partial x}\right) = \mu\frac{\partial}{\partial x}\left[\nabla^2\Gamma - 2\alpha\frac{\partial^2\Gamma}{\partial z^2}\right] \tag{24}$$
$$\tau_{yz} = \mu\frac{\partial}{\partial y}\left[\nabla^2\Gamma - 2\alpha\frac{\partial^2\Gamma}{\partial z^2}\right]$$

Note that if the stress quantity in brackets is zero, then the stress terms will be zero.

We then have $\Gamma(\mathbf{k},z) = (A + B\beta z)e^{\beta z} \qquad \Gamma(\mathbf{k},0) = A \qquad \left.\frac{\partial^2\Gamma}{\partial z^2}\right|_0 = \beta^2(2B + A)$

where $\beta = 2\pi|\mathbf{k}|$, which we will substitute into τ_{xz},

$$\beta^2\left[-A + (1-2\alpha)(2B + A)\right] = 0 \tag{25}$$
$$A = -B\left(2 - \frac{1}{\alpha}\right)$$

We then make a substitution for A, which reduces the Galerkin vector and its derivatives to

$$\Gamma(\mathbf{k},z) = B\left(\beta z + \frac{1}{\alpha} - 2\right)e^{\beta z} \qquad \frac{\partial^2\Gamma}{\partial z^2} = \beta^2 B\left[\beta z + \frac{1}{\alpha}\right]e^{\beta z} \ . \tag{26}$$

We pause here briefly to summarize our solutions thus far,

$$\Gamma(\mathbf{k},z) = -B\left(2 - \frac{1}{\alpha} - \beta z\right)e^{\beta z}$$
$$U_B = -i\alpha(2\pi)k_x\beta B\left(1 - \frac{1}{\alpha} - \beta z\right)e^{\beta z}$$
$$V_B = -i\alpha(2\pi)k_y\beta B\left(1 - \frac{1}{\alpha} - \beta z\right)e^{\beta z} \tag{27}$$
$$W_B = -\alpha\beta^2 B\left(\frac{1}{\alpha} - \beta z\right)e^{\beta z}$$
$$\tau_{xx} = 2\beta B\left\{\mu\alpha(2\pi k_x)^2 + \lambda(1-\alpha)(2\pi k_y)^2 + \mu\alpha(2\pi k_x)^2\beta z\right\}e^{\beta z}$$
$$\tau_{yy} = 2\beta B\left\{\mu\alpha(2\pi k_y)^2 + \lambda(1-\alpha)(2\pi k_x)^2 + \mu\alpha(2\pi k_y)^2\beta z\right\}e^{\beta z}$$
$$\tau_{zz} = -2\alpha\mu(\beta)^3 B(1 - \beta z)e^{\beta z}$$

Our solutions differ from those in the *Steketee* paper because we are solving for a solution at z < 0.

Now use τ_{zz} to solve for B:

$$\begin{aligned}\tau_{zz} &= -2\alpha\mu B\beta^3[1-\beta z]e^{\beta z} \\ \tau_{zz}(\mathbf{k},0) &= -2\alpha\mu B\beta^3 = -\tau_3\end{aligned} \tag{28}$$

Solving for B, $\quad \mathrm{B}(\mathbf{k}) = \dfrac{\tau_3(\mathbf{k})}{\mu\beta^3}$

We then make the appropriate substituions for B and find our final Boussinesq solutions where dW_B / dz is needed for the stress computation.

$$\begin{aligned}U_{\mathrm{B}} &= -i2\pi k_x \frac{1}{2\mu}\frac{\tau_3(\mathbf{k})}{\beta^2}\left[1-\frac{1}{\alpha}-\beta z\right]e^{\beta z} \\ V_{\mathrm{B}} &= -i2\pi k_y \frac{1}{2\mu}\frac{\tau_3(\mathbf{k})}{\beta^2}\left[1-\frac{1}{\alpha}-\beta z\right]e^{\beta z} \\ W_{\mathrm{B}} &= -\frac{1}{2\mu}\frac{\tau_3(\mathbf{k})}{\beta}\left[\frac{1}{\alpha}-\beta z\right]e^{\beta z} \\ \frac{dW_B}{dz} &= -\frac{\tau_3(\mathbf{k})}{2\mu}\left[\frac{1}{\alpha}-1-\beta z\right]e^{\beta z}\end{aligned} \tag{29}$$

$$\text{where } \tau(\mathbf{k}) = e^{\beta a}\begin{bmatrix} -i\dfrac{k_x}{|k|}\left(\alpha\beta a + \dfrac{\mu}{(\lambda+2\mu)}\right)F_x \\ -i\dfrac{k_y}{|k|}\left(\alpha\beta a + \dfrac{\mu}{(\lambda+2\mu)}\right)F_y \\ (\alpha\beta a - 1)F_z \\ \end{bmatrix} \tag{30}$$

To re-cap, we now have three solutions that comprise our total solution:

$$\begin{bmatrix} U(\mathbf{k}) \\ V(\mathbf{k}) \\ W(\mathbf{k}) \end{bmatrix} = \begin{bmatrix} U(\mathbf{k}) \\ V(\mathbf{k}) \\ W(\mathbf{k}) \end{bmatrix}^{source} + \begin{bmatrix} U(\mathbf{k}) \\ V(\mathbf{k}) \\ W(\mathbf{k}) \end{bmatrix}^{image} + \begin{bmatrix} U_B(\mathbf{k}) \\ V_B(\mathbf{k}) \\ W_B(\mathbf{k}) \end{bmatrix} \tag{31}$$

The following Matlab code verifies Boussinesq solution:

```
%  matlab routine to check the algebra for the Boussinesq solution from
%  Steketee, J. A., On Volterra's dislocations in a semi-infinite elastic
%  medium, Can. J. Phys.,  V. 36, p. 192-205, 1958.
%
       pi = sym('pi');
       pi2= 4*pi*pi;
%  wavenumbers and observation depth z
       kx=sym('kx');
       ky=sym('ky');
       kh2=kx*kx+ky*ky;
       kh=sqrt(kh2);
       lam=sym('lam');
       mu=sym('mu');
       al=(lam+mu)/(lam+2*mu);
       syms Gam x y z;
       ex=exp(2*pi*kh*z);
%  check Tau_xz/yz for correct algebra
%  define U,V,W as function of tau
       laplace=diff(Gam,x,2) + diff(Gam,y,2) + diff(Gam,z,2);
       U=-al*diff((diff(Gam,x)),z);
       V=-al*diff((diff(Gam,y)),z);
       W=-al*diff(Gam,z,2) + laplace;
%  define tau_xz as in notes
       tauxz=(mu)*(diff(U,z) + diff(W,x));
%  define tauzz as function of Galerkin vector
       simplify_tauxz=mu*diff((laplace - 2*al*diff(Gam,z,2)),x);
%  take difference to find mistakes
       tauzero=simplify(tauxz-simplify_tauxz)

ans:    tauzero  =   0

%  check bracket term for correct algebra
%  unknown constants
       A=sym('A');
       B=sym('B');
       beta=2*pi*kh;
       G=ex*(A+B*beta*z);
%  take the derivatives of the galerkin vector
        Gz = diff(G,'z');
        Gzz= diff(Gz,'z');
%  evaluate at z=0
       G0   = limit(G,'z',0);
       Gz0  = limit(Gz,'z',0);
       Gzz0 = limit(Gzz,'z',0);
%  term in bracket
       Bracket=(((2*pi*i*kx)^2)*G0 + ((2*pi*i*ky)^2)*G0 + Gzz0) - 2*al*Gzz0;
%      solve Bracket for 'A'
       A=solve(Bracket,A);
%  define A as in notes
       myA=-B*(2-1/al);
%  take difference to find mistakes
       Azero=simplify(myA-A)

ans:     Azero  =   0
```

```
%  check new solutions with A substitution
%  evaluate Galerkin vector for solution of A
        G=limit( (ex*(A+B*beta*z)),'A',myA);
%  take the derivatives of the Galerkin vector
        Gz = diff(G,'z');
        Gzz= diff(Gz,'z');
%  check Galerkin, U,V,W,txx,tyy,tzz for mistakes
        Galerkin=-B*(2-1/al-beta*z)*ex;
        zeroGalerkin=simplify(G-Galerkin)

ans:      zeroGalerkin   =   0

        U=-al*2*pi*i*kx*Gz;
        newU=i*al*2*pi*kx*beta*B*(1-1/al-beta*z)*ex;
        zeroU=simplify(U-newU)

ans:      zeroU  =   0

        V=-al*2*pi*i*ky*Gz;
        newV=i*al*2*pi*ky*beta*B*(1-1/al-beta*z)*ex;
        zeroV=simplify(V-newV)

ans:      zeroV  =   0

        W=(1-al)*Gzz + (2*pi*i*kx)^2*G + (2*pi*i*ky)^2*G;
        newW=al*beta^2*B*(1/al-beta*z)*ex;
        zeroW=simplify(W-newW)

ans:      zeroW  =   0

        txx= lam*( (2*pi*i*kx)*U + (2*pi*i*ky)*V + diff(W,z)) + 2*mu*(2*pi*i*kx*U);
        newtxx= 2*beta*B*(mu*al*(2*pi*kx)^2 + lam*(1-al)*(2*pi*ky)^2 +
               mu*al*(2*pi*kx)^2*beta*z)*ex;
        zerotxx=simplify(txx-newtxx)

ans:      zerotxx   =   0

        tyy= lam*( (2*pi*i*kx)*U + (2*pi*i*ky)*V + diff(W,z)) + 2*mu*(2*pi*i*ky*V);
        newtyy= 2*beta*B*(mu*al*(2*pi*ky)^2 + lam*(1-al)*(2*pi*kx)^2 +
               mu*al*(2*pi*ky)^2*beta*z)*ex;
        zerotyy=simplify(tyy-newtyy)

ans:      zerotyy   =   0

        tzz=lam*( (2*pi*i*kx)*U + (2*pi*i*ky)*V + diff(W,z)) + 2*mu*(diff(W,z));
        newtzz=2*al*mu*beta^3*B*(1-beta*z)*ex;
        zerotzz=simplify(tzz-newtzz)

ans:      zerotzz   =   0


%  sum of derivatives of txz, tyz, tzz should be zero if solutions satisfy the differential
   equation
        txz = mu*(diff(newU,'z')+i*2*pi*kx*newW);
        tyz = mu*(diff(newV,'z')+i*2*pi*ky*newW);
        Oursum= simplify(i*2*pi*kx*txz + i*2*pi*ky*tyz + diff(tzz,'z'))

ans:      Oursum  =   0
```

```
%  check the Stekeete Solution
       Sex=exp(-2*pi*kh*z)/((2*pi*kh)^2);
       SU=Sex*i*2*pi*kx*(2*pi*kh*z-1/al+1);
       SV=Sex*i*2*pi*ky*(2*pi*kh*z-1/al+1);
       SW=-Sex*2*pi*kh*(2*pi*kh*z+1/al);
       STxz=mu*(diff(SU,'z')+i*2*pi*kx*SW);
       STyz=mu*(diff(SV,'z')+i*2*pi*ky*SW);
       STzz=(lam+2*mu)*diff(SW,'z')+lam*(i*2*pi*kx*SU+i*2*pi*ky*SV);
       Steketeesum= simplify(i*2*pi*kx*STxz+i*2*pi*ky*STyz+diff(STzz,'z'))

ans:     Steketeesum   =   0


%  check final Boussinesq solutions
       syms t3
       newB = -t3/(2*al*mu*beta^3);
%  evaluate U,V,W at new solution for B, take difference in notes to find any mistakes
       UB1=limit(U,'B',newB);
       myUB1 = -i*2*pi*kx/(2*mu)*t3/(beta^2)*(1-1/al-beta*z)*ex;
       zeroUB1=simplify(UB1-myUB1)

ans:     zeroUB1   =   0

       VB1=limit(V,'B',newB);
       myVB1 = -i*2*pi*ky/(2*mu)*t3/(beta^2)*(1-1/al-beta*z)*ex;
       zeroVB1=simplify(VB1-myVB1)


ans:     zeroVB1   =   0

       WB1=limit(W,'B',newB);
       myWB1 = -1/(2*mu)*t3/beta*(1/al-beta*z)*ex;
       zeroWB1=simplify(WB1-myWB1)

ans:     zeroWB1   =    0
```

The following Matlab code verifies that the original differential equation is satisfied with the Boussinesq contribution:

```
%
%  matlab routine to compute stresses (tau(z)) from  the body-force solution for the
   source, the image, and the boussinesq correction

%  first part checks to see if the differential equation is satisfied
%  second part incorporates boussinesq correction, solves for new stress tensor, and
   computes a zero stress tensor
%
       pi = sym('pi');
       pi2= 4*pi*pi;
%  wavenumbers and observation depth z
       kx=sym('kx');
       ky=sym('ky');
        kh2=kx*kx+ky*ky;
        kh=sqrt(kh2);
```

```
%  source depth and elastic constants
    a=sym('a');
    z =sym('z');
    lam=sym('lam');
    mu=sym('mu');
%  solution in Fourier domain after z-integration
    Bs=exp(-2*pi*kh*(z-a))/(2*pi*kh);
    Bi=exp(-2*pi*kh*(-z-a))/(2*pi*kh);
    C=(lam+mu)/(4*mu*(lam+2*mu));
    D=(lam+3*mu)/(lam+mu);
%  constants
    alpha = (lam + mu)/(lam+ 2*mu);
    beta = 2*pi*kh;
    ab1=(1/alpha + beta.*z);
    ab2=(1/alpha + beta.*z + 1);
    ez=exp(beta.*z);
    A = exp(beta.*a);
    B = alpha*beta.*a;
    MUU= mu/(lam+2*mu);
%  generate compents of stress tensor
        t1x = 0*A;
        t1y = 0*A;
        t1z = 0*A;
        t2x = 0*A;
        t2y = 0*A;
        t2z = 0*A;
        t3x = -i*(B+MUU).*kx./kh.*A;
        t3y = -i*(B+MUU).*ky./kh.*A;
        t3z = (B-1)*A;
%  x-component
    Uxs =  Bs*C*(D + (ky*ky)/kh2 - 2*pi*kh*(z-a)*kx*kx/kh2);
    Uxi =  Bi*C*(D + (ky*ky)/kh2 - 2*pi*kh*(-z-a)*kx*kx/kh2);
    Ubx = -2*pi*i/(2*mu)*kx.*t3x./(beta.*beta).*(1-ab1).*ez;
    Ux  =  Uxs + Uxi + Ubx;
    Uys = -Bs*C*kx*ky*(1+2*pi*kh*(z-a))/kh2;
    Uyi = -Bi*C*kx*ky*(1+2*pi*kh*(-z-a))/kh2;
    Uby = -2*pi*i/(2*mu)*kx.*t3y./(beta.*beta).*(1-ab1).*ez;
    Uy  =  Uys + Uyi + Uby;
     Uzs = -Bs*C*i*2*pi*kx*(z-a);
    Uzi = -(Bi*C*i*2*pi*kx*(-z-a));   %make z image negative
    Ubz = -2*pi*i/(2*mu)*kx.*t3z./(beta.*beta).*(1-ab1).*ez;
    Uz  =  Uzs + Uzi + Ubz;
%  y-component
    Vbx = -2*pi*i/(2*mu)*ky.*t3x./(beta.*beta).*(1-ab1).*ez;
    Vx  =  Uys + Uyi + Vbx ;
    Vys =  Bs*C*(D + (kx*kx)/kh2 - 2*pi*kh*(z-a)*ky*ky/kh2);
    Vyi =  Bi*C*(D + (kx*kx)/kh2 - 2*pi*kh*(-z-a)*ky*ky/kh2);
    Vby = -2*pi*i/(2*mu)*ky.*t3y./(beta.*beta).*(1-ab1).*ez;
    Vy  =  Vys + Vyi + Vby;
     Vzs = -Bs*C*i*2*pi*ky*(z-a);
    Vzi = -(Bi*C*i*2*pi*ky*(-z-a));    %make z image negative
     Vbz = -2*pi*i/(2*mu)*ky.*t3z./(beta.*beta).*(1-ab1).*ez;
     Vz  =  Vzs + Vzi + Vbz ;
%  z-component
    Wxs = -Bs*C*i*2*pi*kx*(z-a);
    Wxi =  Bi*C*i*2*pi*kx*(-z-a);
    Wbx = -1/(2*mu)*t3x./(beta).*(1/alpha - beta.*z).*ez;
    Wx  =  Wxs + Wxi + Wbx;
    Wys = -Bs*C*i*2*pi*ky*(z-a);
     Wyi =  Bi*C*i*2*pi*ky*(-z-a);
    Wby=  -1/(2*mu)*t3y./(beta).*(1/alpha - beta.*z).*ez;
    Wy  =  Wys + Wyi + Wby;
    Wzs =  Bs*C*(D + 2*pi*kh*(z-a));
    Wzi = -(Bi*C*(D + 2*pi*kh*(-z-a)));
    Wbz = -1/(2*mu)*t3z./(beta).*(1/alpha - beta.*z).*ez;
    Wz  =  Wzs + Wzi + Wbz;
%  add together to check differential equation
    U = Ux + Uy + Uz;
    V = Vx + Vy + Vz;
    W = Wx + Wy + Wz;
%  compute rhx
    RHX = mu*(-pi2*kx*kx*U-pi2*ky*ky*U+diff(U,'z',2));
    RHX = simplify(RHX + (lam+mu)*(-pi2*kx*kx*U-pi2*kx*ky*V+i*2*pi*kx*diff(W,'z',1)))
%  compute rhy
    RHY = mu*(-pi2*kx*kx*V-pi2*ky*ky*V+diff(V,'z',2));
    RHY = simplify(RHY + (lam+mu)*(-pi2*kx*ky*U-pi2*ky*ky*V+i*2*pi*ky*diff(W,'z',1)))
% compute rhz
    RHZ = mu*(-pi2*kx*kx*W-pi2*ky*ky*W+diff(W,'z',2));
    RHZ = simplify(RHZ +
         (lam+mu)*(i*2*pi*kx*diff(U,'z',1)+i*2*pi*ky*diff(V,'z',1)+diff(W,'z',2)))

ans:   RHX  =    0
       RHY  =    0
       RHZ  =    0

%  now compute the derivative of the displacements with respect to z
    Uxz = diff(Ux,'z');
    Uyz = diff(Uy,'z');
    Uzz = diff(Uz,'z');
    Vxz = diff(Vx,'z');
    Vyz = diff(Vy,'z');
    Vzz = diff(Vz,'z');
    Wxz = diff(Wx,'z');
    Wyz = diff(Wy,'z');
    Wzz = diff(Wz,'z');
%  now compute the stresses, tau-xz  tau-yz  tau-z
  T11=mu*(limit(Uxz+i*2*pi*kx*Wx,'z',0));
  T12=mu*(limit(Uyz+i*2*pi*kx*Wy,'z',0));
  T13=mu*(limit(Uzz+i*2*pi*kx*Wz,'z',0));
  T21=mu*(limit(Vxz+i*2*pi*ky*Wx,'z',0));
  T22=mu*(limit(Vyz+i*2*pi*ky*Wy,'z',0));
  T23=mu*(limit(Vzz+i*2*pi*ky*Wz,'z',0));
  T31=(lam + 2*mu)*limit(Wxz,'z',0)+lam*limit(i*2*pi*kx*Ux+i*2*pi*ky*Vx,'z',0);
  T32=(lam + 2*mu)*limit(Wyz,'z',0)+lam*limit(i*2*pi*kx*Uy+i*2*pi*ky*Vy,'z',0);
  T33=(lam + 2*mu)*limit(Wzz,'z',0)+lam*limit(i*2*pi*kx*Uz+i*2*pi*ky*Vz,'z',0);
%  new stress tensor
Tauz=simplify([T11, T12, T13;
       T21, T22, T23;
       T31, T32, T33]);
newTXZ = simplify([T11 T12 T13])
newTYZ = simplify([T21 T22 T23])
newTZZ = simplify([T31 T32 T33])


ans:   newTXZ  =   [ 0,  0,  0]
       newTYZ  =   [ 0,  0,  0]
       newTZZ  =   [ 0,  0,  0]
```

vii) ***Screw dislocation and numerical test with line source***

We will now construct a screw dislocation source by taking the derivative of the point source in the direction normal to the fault plane. This corresponds to multiplication in the Fourier transform domain.

$$\begin{aligned} &\text{E - W trending fault: } \Im\left[\frac{df(x,y)}{dy}\right] = i2\pi k_y F(k_x,k_y) \\ &\text{N - S trending fault: } \Im\left[\frac{df(x,y)}{dx}\right] = i2\pi k_x F(k_x,k_y) \end{aligned} \tag{32}$$

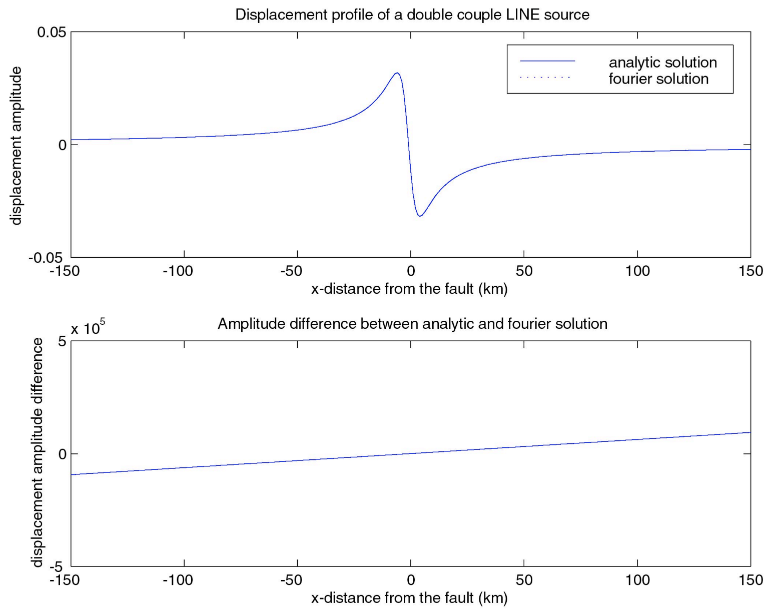
In practice, the body forces due to the stress discontinuity across a fault plane are approximated by the derivative of a Gaussian function, effectively producing a model fault with a finite thickness. Curved faults are constructed with short line segments having cosine tapered ends and are typically 6-10 km long. The fault trace is imbedded in a two-dimensional grid which is Fourier transformed, multiplied by the transfer functions above, and inverse Fourier transformed. The horizontal complexity of the fault system has no effect on the speed of the computation. However, variations in locking depth along the fault system require computing the model for each different locking depth and the outputs summed to form the full solution.

In a previous derivation (http://topex.ucsd.edu/geodynamics/13fault_disp.pdf) we showed that the *y*-displacement *v* due to a line source is given by

$$v(x) = \frac{-A}{\pi\mu}\frac{x}{\left(x^2 + a^2\right)} \tag{33}$$

where *A* is the line force per unit length. For comparison with this solution we must divide the fourier solution by Δx which corresponds to the cell-size in the *x*-direction. The comparison between the two solutions is shown in Figure 1. For a source depth of 10 km, the *x*-length of the fourier grid must be greater than 8192 to achieve better than 10^{-3} numerical accuracy. A smaller dimension can be used if the fault has a finite length in the *y*-direction.

Figure 1



The following Matlab code was used to compute and display the above analytical comparison.

```
%matlab program to find displacement field due to line source
%
%
%  set up line source geometry
       depth=-10;     % depth (km)
       z=0;           % observation depth
%  lame parameters
       lam=1;
       mu=1;
       alpha = (lam + mu)/(lam+2*mu);
%  generate wave numbers in x direction
       Lx=512;               % length
       nx=512;               % number of pixels in x
       dx=Lx/nx;             % spacing in x
       nx2=nx/2;
       kx= -nx2: (nx2-1);    % set up x wave number vector
       kx= kx./Lx;           % wave numbers in x
%  generate wave numbers in y direction
       Ly=512;               % length
       ny=512;               % number of pixels in y
       dy=Ly/ny;             % spacing in y
       ny2=ny/2;
       ky= -ny2 : (ny2-1);   % set up y wave number vector
       ky= ky'./Ly;          % wave numbers in y
%  set up fault azimuth
       theta=90.;
       rad=pi/180.;
       thetas=theta*rad;
       thetan=thetas+pi/2.
%  set up pt y-source force in center of grid
       yy=(1:ny)';
       xx=nx/2*ones(length(yy),1);
       force=ones(length(yy),1);
```

```
%  set up force option
       px=zeros(ny,nx);
       py=zeros(ny,nx);
       pz=zeros(ny,nx);
       po=zeros(ny,nx);
   for t=1:length(xx)
       po(yy(t),xx(t))=force(t);
       end
       py=po.*sin(thetas);
       px=po.*cos(thetas);
       pz=po;
%  fourier transform pt force grid into k domain
       pkx=fftshift(fft2(px));
       pky=fftshift(fft2(py));
       pkz=fftshift(fft2(pz));
%  generate the 2-D wavenumber arrays
       [kxx,kyy]=karray(kx,ky);
%  generate model for source and image [sU,sV,sW], [iU,iV,iW]
       [sUx,sVx,sWx,sUy,sVy,sWy,sUz,sVz,sWz]=sourceU(kxx,kyy,depth,lam,mu,z);
       [iUx,iVx,iWx,iUy,iVy,iWy,iUz,iVz,iWz]=imageU(kxx,kyy,depth,lam,mu,z);
%  add the source-solutions
       sU = sUx + sUy + sUz;
       sV = sVx + sVy + sVz;
       sW = sWx + sWy + sWz;
%  add the image-solutions
       iU = iUx + iUy + iUz;
       iV = iVx + iVy + iVz;
       iW = iWx + iWy + iWz;
%  add both source and image solutions together
       Usi = sU + iU;
       Vsi = sV + iV;
       Wsi = sW + iW;
%  generate tau solutions (stess in z)
       [t1x,t1y,t1z,t2x,t2y,t2z,t3x,t3y,t3z]=tauz(kxx,kyy,depth,lam,mu);
%  generate the boussinesq solution :
       [Ub,Vb,Wb]=boussinesq(kxx,kyy,lam,mu,t3x,t3y,t3z,z);
%  add the boussinesq solutions to the source-image solution:
       U = Usi + Ub;
       V = Vsi + Vb;
       W = Wsi + Wb;
%  generate force couple
       k = cos(thetan).*kxx + sin(thetan).*kyy  ;
       Vcouple = -2*pi*i.*k.*V.*pky ;
%  inverse fft for integrated source
       v = real(ifft2(fftshift(Vcouple)));
       vfourier=real(v(ny2,:));
%  set up analytic solution (Weertman)
       xp=-(nx2-1):1:nx2;
       xp=xp*dx;
       vanalytical=-1*(dx.*xp./(xp.*xp+(depth.*depth))/(pi*mu));
%  set up x and y axis for image
       xd= (0:dx:Lx)';
        yd= -(0:dy:Ly)';
%  display
       subplot(2,1,1),plot(axisx,vanalytical,'b',axisx,vfourier,':')
       legend('analytic solution', 'fourier solution')
       title('Displacement profile of a double couple LINE source')
       xlabel('x-distance from the fault (km)')
       ylabel('displacement amplitude ')
       vdiff = vfourier - vanalytical;
       subplot(2,1,2),plot(axisx,vdiff)
       title('Amplitude difference between analytic and fourier solution')
       xlabel('x-distance from the fault (km)')
       ylabel('displacement amplitude difference')
```

viii) ***Vertical integration of point source to create fault plane and numerical test***

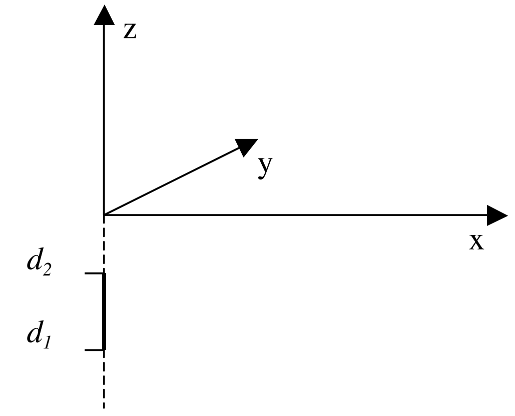
We integrate the point source Green's function between depths d_1 and d_2 to simulate a fault plane. For a complex dipping fault, this integration could be done numerically. However, if the faults are assumed to be vertical, the integration can be performed analytically. The body force is applied between the deep depth d_1 (e.g., minus infinity) and the shallow depth d_2. The displacement or stress (derivatives are computed analytically) can be evaluated at any depth z above d_2. Note the displacement is the sum of three terms: a source, an image, and a Boussinesq correction.

For this exercise, we'll assume $z = 0$. The integration will be over the source depth, a.

Let $z' = a$. An examination of the solution above shows that there are only two basic integrals to perform.

$$(1) = \left(2\pi|k|\right)^{-1}\int_{d_1}^{d_2} e^{2\pi|k|z'}dz'$$

$$(2) = \int_{d_1}^{d_2} z'\, e^{2\pi|k|z'}dz'$$



The first integral is

$$(1) = \left(2\pi|k|\right)^{-2}\left(e^{2\pi|k|d_2} - e^{2\pi|k|d_1}\right) \tag{34}$$

The second integral is performed using integration by parts

$$(2) = \left(2\pi|k|\right)^{-2}\left(e^{2\pi|k|d_2}\left(2\pi|k|d_2 - 1\right) - e^{2\pi|k|d_1}\left(2\pi|k|d_1 - 1\right)\right) \tag{35}$$

After a little algebra, the solution matrix for both source and image is

$$\begin{bmatrix} U(\mathbf{k}) \\ V(\mathbf{k}) \\ W(\mathbf{k}) \end{bmatrix}^{source} = \begin{bmatrix} U_{xs} & U_{ys} & U_{zs} \\ U_{ys} & V_{ys} & V_{zs} \\ U_{zs} & V_{zs} & W_{zs} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \qquad \begin{bmatrix} U(\mathbf{k}) \\ V(\mathbf{k}) \\ W(\mathbf{k}) \end{bmatrix}^{image} = \begin{bmatrix} U_{xi} & U_{yi} & -^{*}U_{zi} \\ U_{yi} & V_{yi} & -^{*}V_{zi} \\ U_{zi} & V_{zi} & -^{*}W_{zi} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \tag{36}$$

where the souce solutions are:

$$U_{xs}(\mathbf{k}) = \frac{C}{\beta^2}\left\{e^{-\beta(z-d_2)}\left[D + \frac{k_y^2}{|\mathbf{k}|^2} - \frac{k_x^2}{|\mathbf{k}|^2}\left(1+\beta(z-d_2)\right)\right] - e^{-\beta(z-d_1)}\left[D + \frac{k_y^2}{|\mathbf{k}|^2} - \frac{k_x^2}{|\mathbf{k}|^2}\left(1+\beta(z-d_1)\right)\right]\right\}$$

$$U_{ys}(\mathbf{k}) = -\frac{C}{\beta^2}\frac{k_x k_y}{|\mathbf{k}|^2}\left\{e^{-\beta(z-d_2)}\left(2+\beta(z-d_2)\right) - e^{-\beta(z-d_1)}\left(2+\beta(z-d_1)\right)\right\}$$

$$U_{zs}(\mathbf{k}) = -i\frac{C}{\beta^2}\frac{k_x}{|\mathbf{k}|}\left\{e^{-\beta(z-d_2)}\left(1+\beta(z-d_2)\right) - e^{-\beta(z-d_1)}\left(1+\beta(z-d_1)\right)\right\}$$

$$V_{ys}(\mathbf{k}) = \frac{C}{\beta^2}\left\{e^{-\beta(z-d_2)}\left[D + \frac{k_x^2}{|\mathbf{k}|^2} - \frac{k_y^2}{|\mathbf{k}|^2}\left(1+\beta(z-d_2)\right)\right] - e^{-\beta(z-d_1)}\left[D + \frac{k_x^2}{|\mathbf{k}|^2} - \frac{k_y^2}{|\mathbf{k}|^2}\left(1+\beta(z-d_1)\right)\right]\right\}$$

$$V_{zs}(\mathbf{k}) = -i\frac{C}{\beta^2}\frac{k_y}{|\mathbf{k}|}\left\{e^{-\beta(z-d_2)}\left(1+\beta(z-d_2)\right) - e^{-\beta(z-d_1)}\left(1+\beta(z-d_1)\right)\right\}$$

$$W_{zs}(\mathbf{k}) = \frac{C}{\beta^2}\left\{e^{-\beta(z-d_2)}\left[D+1+\beta(z-d_2)\right] - e^{-\beta(z-d_1)}\left[D+1+\beta(z-d_1)\right]\right\} \qquad (37)$$

and the image solutions are:

$$U_{xi}(\mathbf{k}) = \frac{C}{\beta^2}\left\{e^{\beta(z+d_2)}\left[D + \frac{k_y^2}{|\mathbf{k}|^2} - \frac{k_x^2}{|\mathbf{k}|^2}\left(1-\beta(z+d_2)\right)\right] - e^{\beta(z+d_1)}\left[D + \frac{k_y^2}{|\mathbf{k}|^2} - \frac{k_x^2}{|\mathbf{k}|^2}\left(1-\beta(z+d_1)\right)\right]\right\}$$

$$U_{yi}(\mathbf{k}) = -\frac{C}{\beta^2}\frac{k_x k_y}{|\mathbf{k}|^2}\left\{e^{\beta(z+d_2)}\left(2-\beta(z+d_2)\right) - e^{\beta(z+d_1)}\left(2-\beta(z+d_1)\right)\right\}$$

$$^{*}U_{zi}(\mathbf{k}) = i\frac{C}{\beta^2}\frac{k_x}{|\mathbf{k}|}\left\{e^{\beta(z+d_2)}\left(1-\beta(z+d_2)\right) - e^{\beta(z+d_1)}\left(1-\beta(z+d_1)\right)\right\}$$

$$V_{yi}(\mathbf{k}) = \frac{C}{\beta^2}\left\{e^{\beta(z+d_2)}\left[D + \frac{k_x^2}{|\mathbf{k}|^2} - \frac{k_y^2}{|\mathbf{k}|^2}\left(1-\beta(z+d_2)\right)\right] - e^{\beta(z+d_1)}\left[D + \frac{k_x^2}{|\mathbf{k}|^2} - \frac{k_y^2}{|\mathbf{k}|^2}\left(1-\beta(z+d_1)\right)\right]\right\}$$

$$^{*}V_{zi}(\mathbf{k}) = i\frac{C}{\beta^2}\frac{k_y}{|\mathbf{k}|}\left\{e^{\beta(z+d_2)}\left(1-\beta(z+d_2)\right) - e^{\beta(z+d_1)}\left(1-\beta(z+d_1)\right)\right\}$$

$$^{*}W_{zi}(\mathbf{k}) = \frac{C}{\beta^2}\left\{e^{\beta(z+d_2)}\left[D+1-\beta(z+d_2)\right] - e^{\beta(z+d_1)}\left[D+1-\beta(z+d_1)\right]\right\} \qquad (38)$$

where $C = \frac{(\lambda+\mu)}{4\mu(\lambda+2\mu)}$ $\quad D = \frac{\lambda+3\mu}{\lambda+\mu}$ $\quad \alpha = \frac{\lambda+\mu}{\lambda+2\mu}$ $\quad |\mathbf{k}| = \left(k_x^2 + k_y^2\right)^{1/2}$ $\quad \beta = 2\pi|\mathbf{k}|$

* denotes components that are multiplied by a negative sign when computing.

Note that now **F** has units of force/length since we have integrated over depth.

We can rewrite these expressions to isolate the factors containing $e^{-\beta Z}$ and $-\beta Z e^{-\beta Z}$. This will help to simplify the computer code as well as to prepare us for the derivation of the layered models. The solution really contains only 6 independent functions because the tensor is symmetric.

$$\mathbf{U}^s(\mathbf{k},Z) = \begin{bmatrix} U_x & U_y & U_z \\ U_y & V_y & V_z \\ U_z & V_z & W_z \end{bmatrix} \quad \text{and} \quad \mathbf{U}^i(\mathbf{k},Z) = \begin{bmatrix} U_x & U_y & U_z \\ U_y & V_y & V_z \\ -U_z & -V_z & -W_z \end{bmatrix} \tag{39}$$

The 6 independent functions are:

$$\begin{bmatrix} U_x \\ U_y \\ U_z \\ V_y \\ V_z \\ W_z \end{bmatrix} = \frac{C}{\beta^2} \begin{bmatrix} D + \frac{k_y^2}{|\mathbf{k}|^2} - \frac{k_x^2}{|\mathbf{k}|^2} & -\frac{k_x^2}{|\mathbf{k}|^2} \\ -2\frac{k_x k_y}{|\mathbf{k}|^2} & -\frac{k_x k_y}{|\mathbf{k}|^2} \\ -i\frac{k_x}{|\mathbf{k}|} & -i\frac{k_x}{|\mathbf{k}|} \\ D + \frac{k_x^2}{|\mathbf{k}|^2} - \frac{k_y^2}{|\mathbf{k}|^2} & -\frac{k_y^2}{|\mathbf{k}|^2} \\ -i\frac{k_y}{|\mathbf{k}|} & -i\frac{k_y}{|\mathbf{k}|} \\ D+1 & 1 \end{bmatrix} \begin{bmatrix} e^{-\beta|Z|} \\ \beta|Z|e^{-\beta|Z|} \end{bmatrix} \tag{40}$$

The source and image terms can be written as

$$\begin{bmatrix} U(\mathbf{k}) \\ V(\mathbf{k}) \\ W(\mathbf{k}) \end{bmatrix} = \left[\mathbf{U}^s(\mathbf{k},z-d_2) - \mathbf{U}^s(\mathbf{k},z-d_1)\right] \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} + \left[\mathbf{U}^i(\mathbf{k},z+d_2) - \mathbf{U}^i(\mathbf{k},z+d_1)\right] \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \tag{41}$$

Again we can use the symbolic capabilities in Matlab to check these answers. As in the previous case, we insert this solution into the original differential equation where the equations have been Fourier transformed in the *x*- and *y*- directions. The vertical derivatives are carried out symbolically as before.

```
%  matlab routine to check the depth-integrated solution for source space
%
    pi = sym('pi');
    pi2= 4*pi*pi;
%  wavenumbers and observation depth z
    kx=sym('kx');
    ky=sym('ky');
    kh2=kx*kx+ky*ky;
    kh=sqrt(kh2);
%  source depth and elastic constants
    d1=sym('d1');
    d2=sym('d2');
    z =sym('z');
    d1=z-d1;
    d2=z-d2;
    la=sym('la');
    mu=sym('mu');
    lam=la+mu;
%  solution in Fourier domain after z-integration
    C = lam/(4*mu*(la+2*mu));
    D = (la+3*mu)/lam;
    B = 2*pi*kh;
    D1 = B*d1;
    D2 = B*d2;
    E1 = exp(-D1);
    E2 = exp(-D2);
%  compute the x displacements
    t2 = E2*(D + ky*ky/kh2 - kx*kx/kh2*(1+D2));
    t1 = E1*(D + ky*ky/kh2 - kx*kx/kh2*(1+D1));
    Ux = C/(B*B)*(t2-t1);
    Uy = -C/(B*B)*kx*ky/kh2*(E2*(2 + D2) - E1*(2 + D1));
    Uz = -i* C/(B*B)*kx/kh*(E2*(1 + D2) - E1*(1 + D1));
%  compute the y-displacements
    Vx = Uy;
    t2 = E2*(D + kx*kx/kh2 - ky*ky/kh2*(1+D2));
    t1 = E1*(D + kx*kx/kh2 - ky*ky/kh2*(1+D2));
    Vy = C/(B*B)*(t2-t1);
    Vz = -i*C/(B*B)*ky/kh*(E2*(1 + D2) - E1*(1 + D1));
%  compute the z-displacements
    Wx = Uz;
    Wy = Vz;
    Wz = C/(B*B)*(E2*(D + 1 + D2) - E1*(D + 1 + D1));
%  add the solutions
    U = Ux + Uy + Uz;
   V = Vx + Vy + Vz;
    W = Wx + Wy + Wz;
```

```
%  compute rhx
    RHX = mu*(-pi2*kx*kx*U-pi2*ky*ky*U+diff(U,'z',2));
    RHX = RHX + lam*(-pi2*kx*kx*U-pi2*kx*ky*V+i*2*pi*kx*diff(W,'z',1));
    simplify(RHX)
ans:   RHX  =  0
%  compute rhy
    RHY = mu*(-pi2*kx*kx*V-pi2*ky*ky*V+diff(V,'z',2));
    RHY = RHY + lam*(-pi2*kx*ky*U-pi2*ky*ky*V+i*2*pi*ky*diff(W,'z',1));
    simplify(RHY)
```

```
ans:   RHY  =  0
%  compute rhz
    RHZ = mu*(-pi2*kx*kx*W-pi2*ky*ky*W+diff(W,'z',2));
    RHZ = RHZ + lam*(i*2*pi*kx*diff(U,'z',1)+i*2*pi*ky*diff(V,'z',1)+diff(W,'z',2));
    simplify(RHZ)
ans:   RHZ  =  0
```

As with the case of the point source, we also find non-zero terms in the stress tensor corresponding to the normal components:

$$\begin{bmatrix}\tau_{xz}\\ \tau_{yz}\\ \tau_{zz}\end{bmatrix} = \begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix} = \frac{1}{\beta}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \tau_{3x} & \tau_{3y} & \tau_{3z}\end{bmatrix}\begin{bmatrix}F_x\\ F_y\\ F_z\end{bmatrix} \tag{42}$$

$$\begin{aligned}\text{where}\quad \tau_{3x} &= -i\frac{k_x}{|\mathbf{k}|}\left\{e^{\beta d_2}\left(\alpha\beta d_2 - \frac{\lambda}{(\lambda+2\mu)}\right) - e^{\beta d_1}\left(\alpha\beta d_1 - \frac{\lambda}{(\lambda+2\mu)}\right)\right\}\\ \tau_{3y} &= -i\frac{k_y}{|\mathbf{k}|}\left\{e^{\beta d_2}\left(\alpha\beta d_2 - \frac{\lambda}{(\lambda+2\mu)}\right) - e^{\beta d_1}\left(\alpha\beta d_1 - \frac{\lambda}{(\lambda+2\mu)}\right)\right\}\\ \tau_{3z} &= \left\{e^{\beta d_2}\left(\alpha\beta d_2 - \frac{\mu}{(\lambda+2\mu)} - 2\alpha\right) - e^{\beta d_1}\left(\alpha\beta d_1 - \frac{\mu}{(\lambda+2\mu)} - 2\alpha\right)\right\}\end{aligned} \tag{43}$$

The Boussinesq solutions remain the same because they are independent of depth. Therefore, we use the same set of solutions, but simply make a substitution for $\tau_3(k)$ corresponding to our new $\tau_3(k)$ solution. Again we would like to isolate the factors containing $e^{-\beta Z}$ and $-\beta Z e^{-\beta Z}$.

$$\begin{bmatrix}\tau_{3x}\\ \tau_{3y}\\ \tau_{3z}\end{bmatrix} = \mathbf{T}(\mathbf{k},d_2) - \mathbf{T}(\mathbf{k},d_1) \quad \text{where} \quad \begin{bmatrix}T_x\\ T_y\\ T_z\end{bmatrix} = \begin{bmatrix} i\dfrac{\lambda}{(\lambda+2\mu)}\dfrac{k_x}{|\mathbf{k}|} & +i\alpha\dfrac{k_x}{|\mathbf{k}|}\\ i\dfrac{\lambda}{(\lambda+2\mu)}\dfrac{k_y}{|\mathbf{k}|} & +i\alpha\dfrac{k_y}{|\mathbf{k}|}\\ -\dfrac{(2\lambda+3\mu)}{(\lambda+2\mu)} & -\alpha\end{bmatrix}\begin{bmatrix}e^{-\beta|Z|}\\ \beta|Z|e^{-\beta|Z|}\end{bmatrix} \tag{44}$$

Our final integrated solution is then:

$$\begin{bmatrix}U(\mathbf{k})\\ V(\mathbf{k})\\ W(\mathbf{k})\end{bmatrix} = \begin{bmatrix}U(\mathbf{k})\\ V(\mathbf{k})\\ W(\mathbf{k})\end{bmatrix}^{source} + \begin{bmatrix}U(\mathbf{k})\\ V(\mathbf{k})\\ W(\mathbf{k})\end{bmatrix}^{image} + \begin{bmatrix}U_B(\mathbf{k})\\ V_B(\mathbf{k})\\ W_B(\mathbf{k})\end{bmatrix} \tag{45}$$

The following Matlab code verifies this integrated solution, checking both the boundary conditions and differential equation:

```
%  matlab routine to compute stresses (tau(z)) from  the body-force solution for both  source
and the image and Boussinesq correction
%
    pi = sym('pi');
    pi2= 4*pi*pi;
% wavenumbers and observation depth z
    kxx = sym('kxx');
    kyy = sym('kyy');
    kh2 = kxx*kxx+kyy*kyy;
    kh = sqrt(kh2);
% source depth and elastic constants
    syms d1 d2
    z = sym('z');
    lam = sym('lam');
    mu = sym('mu');
    alpha = (lam + mu)/(lam + 2*mu);
    B = 2*pi*kh;
    beta = B;
    C = (lam + mu)/(4*mu*(lam+2*mu));
    D = (lam+3*mu)/(lam + mu);
%  set constants for computing tau solution
       ed1 = exp(beta.*d1);
       Bd1 = beta.*d1;
        ed2 = exp(beta.*d2);
       Bd2 = beta.*d2;
       M2U = mu/(lam+2*mu);
       L2U = lam/(lam+2*mu);
       A1 = -ed1./beta ;
        A2 = -ed2./beta ;
%  generate components of stress tensor
       t1x = 0*A1;
       t1y = 0*A1;
       t1z = 0*A1;
       t2x = 0*A2;
       t2y = 0*A2;
       t2z = 0*A2;
       t3x = A2.*(i*kxx./kh.*(Bd2*alpha-L2U))-A1.*(i*kxx./kh.*(Bd1.*alpha-L2U));
       t3y = A2.*(i*kyy./kh.*(Bd2*alpha-L2U))-A1.*(i*kyy./kh.*(Bd1.*alpha-L2U));
       t3z = A2.*(2*alpha+M2U-alpha*Bd2)-A1.*(2*alpha+M2U-alpha*Bd1);
%  set constants for UVW source and image solutions for upper and lower boundary (2-1)
    es1 = exp(-(z-d1)*B);
    ei1 = exp((z+d1)*B);
    Ms1 = es1./(B.*B);
    Mi1 = ei1./(B.*B);
    es2 = exp(-(z-d2)*B);
    ei2 = exp((z+d2)*B);
    Ms2 = es2./(B.*B);
    Mi2 = ei2./(B.*B);
```

```
%  compute U - displacement
    sUx1 =  Ms1.*C.*(D + kyy.*kyy./kh2 - kxx.*kxx./kh2.*(1 + (z-d1)*B ));
    sUy1 = -Ms1.*C.*(kxx.*kyy./kh2.*(2 + (z-d1)*B));
    sUz1 = -i*Ms1.*C.*kxx./kh.*(1 + (z-d1)*B );
    iUx1 =  Mi1.*C.*(D + kyy.*kyy./kh2 - kxx.*kxx./kh2.*(1 - (z+d1)*B ));
    iUy1 = -Mi1.*C.*(kxx.*kyy./kh2.*(2 - (z+d1)*B));
    iUz1 = -(i*Mi1.*C.*kxx./kh.*(1 - (z+d1)*B));   %negative
         Ux1  =  sUx1 + iUx1;
         Uy1  =  sUy1 + iUy1;
         Uz1  =  sUz1 + iUz1;
         U1 = Ux1 + Uy1 + Uz1;
    sUx2 =  Ms2.*C.*(D + kyy.*kyy./kh2 - kxx.*kxx./kh2.*(1 + (z-d2)*B ));
    sUy2 = -Ms2.*C.*(kxx.*kyy./kh2.*(2 + (z-d2)*B));
    sUz2 = -i*Ms2.*C.*kxx./kh.*(1 + (z-d2)*B );
    iUx2 =  Mi2.*C.*(D + kyy.*kyy./kh2 - kxx.*kxx./kh2.*(1 - (z+d2)*B ));
    iUy2 = -Mi2.*C.*(kxx.*kyy./kh2.*(2 - (z+d2)*B));
    iUz2 = -(i*Mi2.*C.*kxx./kh.*(1 - (z+d2)*B));   %negativ
        Ux2  =  sUx2 + iUx2;
        Uy2  =  sUy2 + iUy2;
        Uz2  =  sUz2 + iUz2;
       U2 = Ux2 + Uy2 + Uz2;
%  compute V - displacement
    sVy1 =  Ms1.*C.*(D + kxx.*kxx./kh2 - kyy.*kyy./kh2.*(1 + (z-d1)*B ));
    sVz1 = -i*Ms1.*C.*kyy./kh.*(1 + (z-d1)*B );
    iVy1 =  Mi1.*C.*(D + kxx.*kxx./kh2 - kyy.*kyy./kh2.*(1 - (z+d1)*B ));
    iVz1 = -(i*Mi1.*C.*kyy./kh.*(1 - (z+d1)*B));    %negative
        Vx1  =  Uy1;
        Vy1  =  sVy1 + iVy1;
        Vz1  =  sVz1 + iVz1;
       V1 = Vx1 + Vy1 + Vz1;
    sVy2 =  Ms2.*C.*(D + kxx.*kxx./kh2 - kyy.*kyy./kh2.*(1 + (z-d2)*B ));
    sVz2 = -i*Ms2.*C.*kyy./kh.*(1 + (z-d2)*B );
    iVy2 =  Mi2.*C.*(D + kxx.*kxx./kh2 - kyy.*kyy./kh2.*(1 - (z+d2)*B ));
    iVz2 = -(i*Mi2.*C.*kyy./kh.*(1 - (z+d2)*B));    %negative
        Vx2  =  Uy2;
        Vy2  =  sVy2 + iVy2;
        Vz2  =  sVz2 + iVz2;
       V2 = Vx2 + Vy2 + Vz2;
%  compute W - displacement
     sWx1 = -i*Ms1.*C.*kxx./kh.*(1 + (z-d1)*B );
     sWy1 = -i*Ms1.*C.*kyy./kh.*(1 + (z-d1)*B );
     sWz1 =  Ms1.*C.*( D + 1 + (z-d1)*B);
     iWx1 =  (i*Mi1.*C.*kxx./kh.*(1 - (z+d1)*B));
     iWy1 =  (i*Mi1.*C.*kyy./kh.*(1 - (z+d1)*B));
     iWz1 = -(Mi1.*C.*( D + 1 - (z+d1)*B));   %negative
       Wx1  = sWx1 + iWx1;
       Wy1  = sWy1 + iWy1;
       Wz1  = sWz1 + iWz1;
       W1 = Wx1 + Wy1 + Wz1;
     sWx2 = -i*Ms2.*C.*kxx./kh.*(1 + (z-d2)*B );
     sWy2 = -i*Ms2.*C.*kyy./kh.*(1 + (z-d2)*B );
     sWz2 =  Ms2.*C.*( D + 1 + (z-d2)*B);
     iWx2 =  (i*Mi2.*C.*kxx./kh.*(1 - (z+d2)*B));
     iWy2 =  (i*Mi2.*C.*kyy./kh.*(1 - (z+d2)*B));
     iWz2 = -(Mi2.*C.*( D + 1 - (z+d2)*B));   %negative
       Wx2  =  sWx2 + iWx2;
       Wy2  =  sWy2 + iWy2;
       Wz2  =  sWz2 + iWz2;
       W2 = Wx2 + Wy2 + Wz2;
```

```
%  subtract lower boundary from upper boundary  (d2-d1)
       Uint = U2 - U1;
       Vint = V2 - V1;
       Wint = W2 - W1;
%  set up boussinesq solutions
       ab = (1/alpha + beta.*z);
       e = exp(beta.*z);
%  generate components of U,V,W
       Ubx = -2*pi*i/(2*mu)*kxx.*t3x./(beta.*beta).*(1-ab).*e;
       Uby = -2*pi*i/(2*mu)*kxx.*t3y./(beta.*beta).*(1-ab).*e;
       Ubz = -2*pi*i/(2*mu)*kxx.*t3z./(beta.*beta).*(1-ab).*e;
       Vbx = -2*pi*i/(2*mu)*kyy.*t3x./(beta.*beta).*(1-ab).*e;
       Vby = -2*pi*i/(2*mu)*kyy.*t3y./(beta.*beta).*(1-ab).*e;
       Vbz = -2*pi*i/(2*mu)*kyy.*t3z./(beta.*beta).*(1-ab).*e;
       Wbx = -1/(2*mu)*t3x./(beta).*(1/alpha - beta.*z).*e;
       Wby = -1/(2*mu)*t3y./(beta).*(1/alpha - beta.*z).*e;
       Wbz = -1/(2*mu)*t3z./(beta).*(1/alpha - beta.*z).*e;
%  add components:
       Ub = Ubx + Uby + Ubz;
       Vb = Vbx + Vby + Vbz;
       Wb = Wbx + Wby + Wbz;
%  add boussinesq solution to source-image solution to check diff. equation
       U = Uint + Ub;
       V = Vint + Vb;
       W = Wint + Wb;
%  check differential equation
%
%  compute rhx
    RHX = mu*(-pi2*kxx*kxx*U-pi2*kyy*kyy*U+diff(U,'z',2));
    RHX = simplify(RHX + (lam+mu)*(-pi2*kxx*kxx*U-pi2*kxx*kyy*V+i*2*pi*kxx*diff(W,'z',1)))
ans:   RHX  =   0
% compute rhy
    RHY = mu*(-pi2*kxx*kxx*V-pi2*kyy*kyy*V+diff(V,'z',2));
    RHY = simplify(RHY + (lam+mu)*(-pi2*kxx*kyy*U-pi2*kyy*kyy*V+i*2*pi*kyy*diff(W,'z',1)))
ans:   RHY  =   0
% compute rhz
    RHZ = mu*(-pi2*kxx*kxx*W-pi2*kyy*kyy*W+diff(W,'z',2));
    RHZ = simplify(RHZ + (lam+mu)*(i*2*pi*kxx*diff(U,'z',1) + i*2*pi*kyy*diff(V,'z',1) +
          diff(W,'z',2))
ans:   RHZ  =   0
%
%  add boussinesq components to source-image components (d2-d1)
       Ux = Ux2 - Ux1 + Ubx;
       Uy = Uy2 - Uy1 + Uby;
       Uz = Uz2 - Uz1 + Ubz;
       Vx = Vx2 - Vx1 + Vbx;
       Vy = Vy2 - Vy1 + Vby;
       Vz = Vz2 - Vz1 + Vbz;
       Wx = Wx2 - Wx1 + Wbx;
       Wy = Wy2 - Wy1 + Wby;
       Wz = Wz2 - Wz1 + Wbz;
%  now compute the derivative of the displacements with respect to z
       Uxz = diff(Ux,'z');
       Uyz = diff(Uy,'z');
       Uzz = diff(Uz,'z');
       Vxz = diff(Vx,'z');
       Vyz = diff(Vy,'z');
       Vzz = diff(Vz,'z');
       Wxz = diff(Wx,'z');
       Wyz = diff(Wy,'z');
       Wzz = diff(Wz,'z');
```

```
%  now compute the stresses, tau-xz  tau-yz  tau-zz
       T11 = mu*(limit(Uxz+i*2*pi*kxx*Wx,'z',0));
       T12 = mu*(limit(Uyz+i*2*pi*kxx*Wy,'z',0));
       T13 = mu*(limit(Uzz+i*2*pi*kxx*Wz,'z',0));
       T21 = mu*(limit(Vxz+i*2*pi*kyy*Wx,'z',0));
       T22 = mu*(limit(Vyz+i*2*pi*kyy*Wy,'z',0));
       T23 = mu*(limit(Vzz+i*2*pi*kyy*Wz,'z',0));
       T31 = (lam + 2*mu)*limit(Wxz,'z',0)+lam*limit(i*2*pi*kxx*Ux+i*2*pi*kyy*Vx,'z',0);
       T32 = (lam + 2*mu)*limit(Wyz,'z',0)+lam*limit(i*2*pi*kxx*Uy+i*2*pi*kyy*Vy,'z',0);
       T33 = (lam + 2*mu)*limit(Wzz,'z',0)+lam*limit(i*2*pi*kxx*Uz+i*2*pi*kyy*Vz,'z',0);
%  if the boussinesq solution is correct, this should be zero
       Tauz = [T11, T12, T13;
               T21, T22, T23;
               T31, T32, T33];
       TXZ = simplify([T11 T12 T13])
       TYZ = simplify([T21 T22 T23])
       TZZ = simplify([T31 T32 T33])

ans:    TXZ  = [ 0,  0,  0]
        TYZ  = [ 0,  0,  0]
        TZZ  = [ 0,  0,  0]
```

We also check this case against an analytic solution. The surface displacement due to a two-dimensional fault that is freely slipping between depths of d_1 and d_2 [*Weertman*, 1964] is:

$$v(x) = \frac{-A}{\pi\mu}\left(\tan^{-1}\frac{x}{d_2} - \tan^{-1}\frac{x}{d_1}\right) \tag{46}$$

where A is slip magnitude.. For comparison with this solution we must divide the Fourier solution by Δx which corresponds to the cell-size in the x-direction. In Figure 2., we have imbedded an infinitely long fault in the y-dimension in a 1-km spaced grid (Δx = 1). We have assigned an upper locking depth of 5 km (d_2) to the fault plane and have extended the lower depth to infinity (d_1). The comparison between the analytic solution (above) and an fault-perpendicular profile of our Fourier model is shown in Figure 3. Numerical accuracy between the analytic and Fourier solutions on the order of 10^{-3} is shown in Figure 4.

Figure 2.

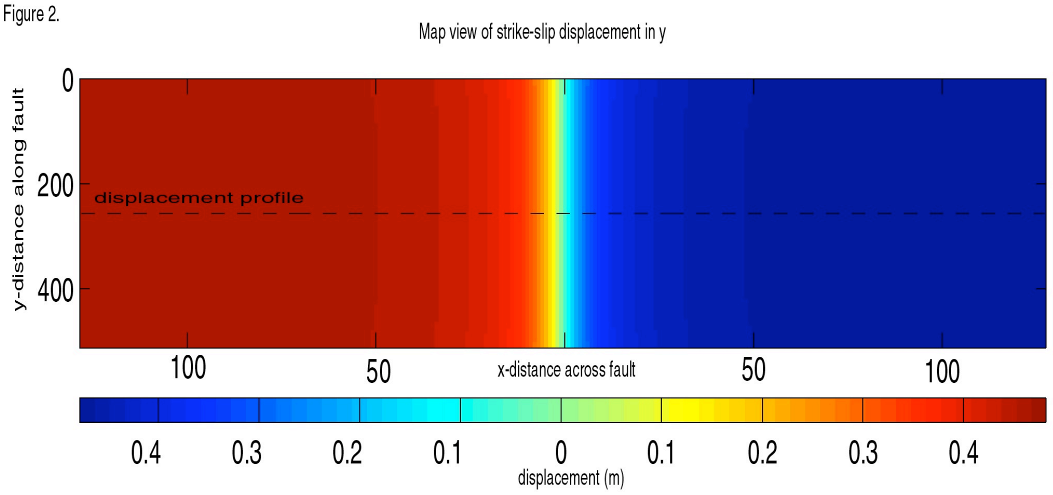


Figure 3

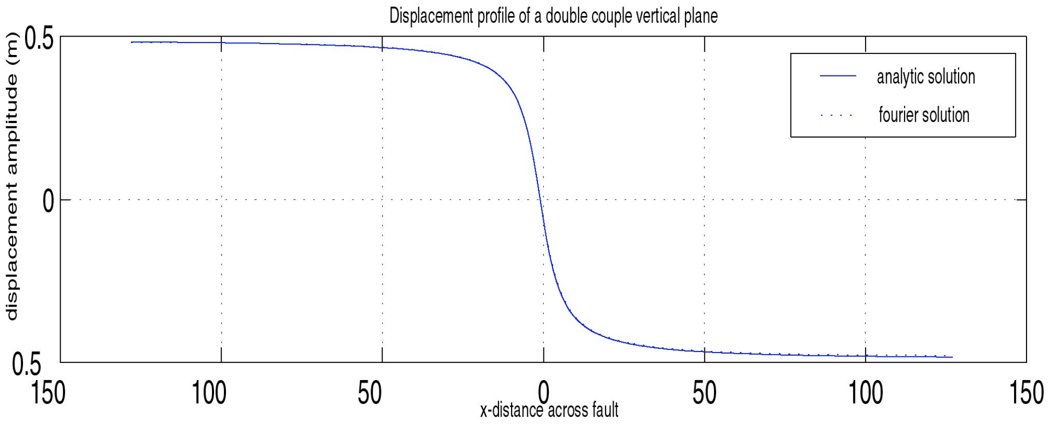
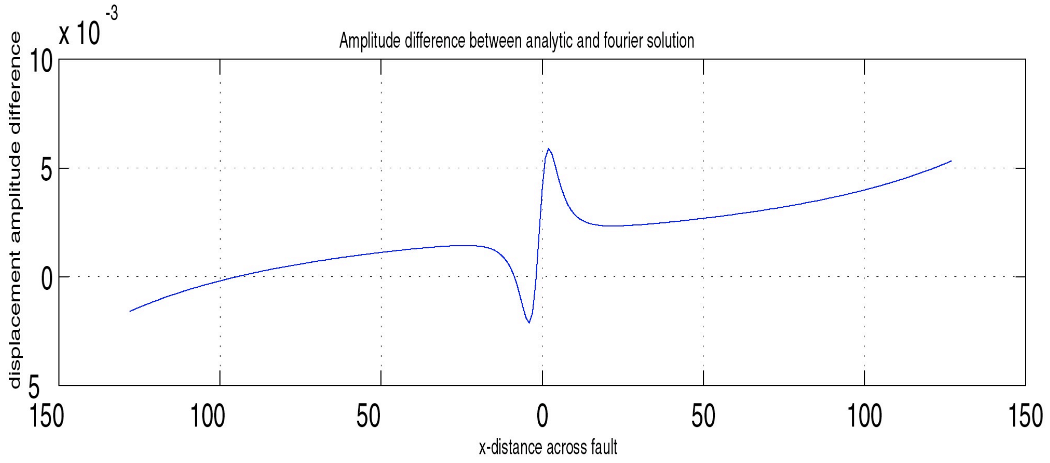


Figure 4.



x) Develop an equivalent body force for a general fault model

Suppose we have a fault plane of finite extent. Assume for now that the fault is vertical and extends from d_1 to d_2 as in the previous model. Let $\mathbf{F}(\mathbf{x})$ be a 3-dimensional body force defined in the fault plane and $\mathbf{n}$ is the normal vector to the plane. Following the displacement formulation of *Okada* [1985], the three components of $\mathbf{F}$ are:

F_1	-	strike-slip displacement (horizontal)
F_2	-	dip-slip displacement (vertical)
F_3	-	opening (horizontal, dike)

Now the three components of the body force are:

$$\begin{aligned} F_x &= F_1 \cos(\theta) - F_3 \sin(\theta) \\ F_y &= F_1 \sin(\theta) + F_3 \cos(\theta) \\ F_z &= F_2 \end{aligned} \tag{47}$$

where θ is the angle between the fault trace and the x-axis. To calculate the equivalent body force vector, we need to take the derivative of the body force vector in the direction that is normal $\mathbf{n}$ to the fault.

$$\rho(\mathbf{x}) = \underline{\underline{\nabla \mathbf{F}(\mathbf{x})}} \bullet \mathbf{n}(\mathbf{x}) \tag{48}$$

Note that $\nabla \mathbf{F}$ is a stress tensor and now has units of force/area. The total displacement vector is

$$V(\mathbf{k}) = \underline{\underline{\mathbf{A}(\mathbf{k})}} \Im_2 \left[\underline{\underline{\nabla \mathbf{F}(\mathbf{x})}} \bullet \mathbf{n}(\mathbf{x}) \right] \tag{49}$$

The $\mathbf{A}$-tensor is the solution provided above relating the vector displacement to the vector body force distribution. As in the case of the Okada models [*Feigl et al.*,1999], one defines a series of fault segments having an orientation, length, and three components of slip. Because our model is a stress formulation rather than a displacement formulation, the three components of slip are replaced by three body forces. The following formula provides a mapping in the case of an infinite fault plane of uniform slip.

$$\mathbf{F} = \mu \Delta V \tag{50}$$

When the lower edge of the fault is extended to infinite depth as in the case of the SAF system model, a Fourier cosine transform is used in the across-fault direction to maintain the far-field velocity $\boldsymbol{V}$ step across the plate boundary. To avoid Fourier artifacts where the fault enters the bottom of the grid and leaves the top of the grid, the fault is extended beyond the top of the model and angled to match the intersection point at the bottom.

The formulation above computes a numerical derivative of the force vector to create a stress tensor using a grid spacing of Δx. This results in the following, more familiar relationship.

$$\boldsymbol{\tau} = \frac{\mathbf{F}}{\Delta x} = \mu \frac{\Delta V}{\Delta x} \tag{51}$$

xi) ***Force couples on a regular grid***

A dislocation on a fault plane is commonly represented by body force couples. In the case of a horizontal strike-slip fault, a double-couple should be used to ensure local balance moment in the horizontal plane [*Burridge and Knopoff,* 1964]. Here we describe the algorithm for generating single- and double-couple body forces for a segmented fault trace mapped onto a regular grid.

Consider a grid with cell spacing *Δx*. The final displacement and stress model cannot resolve features smaller than the cell spacing, thus we approximate a fault segment with a finite length *L* and a finite thickness $\sigma \geq \Delta x$ as follows

$$f(x,y) = g(x)h(y) \tag{52}$$

where the across-fault function *h(y)* is a Gaussian function

$$h(y) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{y^2}{2\sigma^2}\right). \tag{53}$$

We represent a curved fault trace as a large number of straight overlapping segments of the form

$$g(x) = \begin{cases} \left[1 - \cos\dfrac{\pi(x+2\Delta x)}{4\Delta x}\right] & -2\Delta x < x < 2\Delta x \\ 1 & 2\Delta x < x < L-2\Delta x \\ \left[1 - \cos\dfrac{\pi(L+2\Delta x - x)}{4\Delta x}\right] & L-2\Delta x < x < L+2\Delta x \end{cases} \tag{54}$$

where *x* is the distance from the start of the segment and L is the segment length. The segments are arranged end-to-end so the sum of the overlapping cosine functions equals one. The spatial variations in the force-couple are constructed by taking the derivatives of the fault function. The primary couple is parallel to the fault (*x*-direction) and corresponds to the fault-normal derivative

$$f_1(x,y) = g(x)\frac{\partial h}{\partial y} \tag{55}$$

where

$$\frac{\partial h}{\partial y} = \frac{-y}{\sigma^{3/2}\sqrt{2\pi}}\exp\left(\frac{y^2}{2\sigma^2}\right). \tag{56}$$

The secondary force couple is perpendicular to the fault (y-direction) and corresponds to the fault-parallel derivative

$$f_2(x,y) = \frac{\partial g}{\partial x} h(y) \tag{57}$$

where

$$\frac{\partial g}{\partial x} = \begin{cases} \dfrac{\pi}{8\Delta x}\sin\dfrac{\pi(x+2\Delta x)}{4\Delta x} & -2\Delta x < x < 2\Delta x \\ 1 & 2\Delta x < x < L-2\Delta x \\ \dfrac{-\pi}{8\Delta x}\sin\dfrac{\pi(L+2\Delta x - x)}{4\Delta x} & L-2\Delta x < x < L+2\Delta x \end{cases} \quad . \tag{58}$$

A rotation matrix is used to rotate the force couple functions to the proper orientation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \tag{59}$$

where θ is the angle between the x-axis and the fault trace ($\theta > 0$ represents counter-clockwise rotation.

Three modes of displacement can be applied on each fault segment. F_1 is strike-slip, F_2 is dip-slip, and F_3 is opening of the fault. Once the primary and secondary force couple functions are computed and rotated into the fault direction, they are multiplied by the strength of the dislocation to form three grids corresponding to the F_1, F_2, and F_3 modes. These three force components must then be rotated into the Cartesian frame f_x, f_y, and f_z using the following formulas.

$$\begin{pmatrix} f_x(x,y) \\ f_y(x,y) \\ f_z(x,y) \end{pmatrix} = f_1(x,y) \begin{pmatrix} -\cos\theta & & \sin\theta \\ \sin\theta & & \cos\theta \\ & 1 & \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \quad . \tag{60}$$

Balancing of the moment due to the horizontal strike-slip force couple F_1 requires a second force component given by

$$\begin{pmatrix} f_x^2(x,y) \\ f_y^2(x,y) \end{pmatrix} = f_2(x,y) \begin{pmatrix} F_1 \sin\theta \\ F_1 \cos\theta \end{pmatrix} . \tag{61}$$

Note that this force couple only applies to the end of each fault segment and the forces largely cancel where fault segments abut as described in *Burridge and Knopoff* [1964]. The moment generated by the vertical dip-slip F_2 and the opening F_3 of the fault will produce topography that will balance the moment under the restoring force of gravity.

xii) ***Fortran source code for elastic half-space model***

The following Fortran code fftfault.f (and subroutines element.f, fault.f, boussinesq.f) are used to compute displacement and stress from a set of input fault parameters and write these solutions to appropriate grd files. The user sets the fault dimensions, location, and grid spacing from the command line:

```
%
%  fftfault D1 D2 Zobs elements.xyF xd.grd yd.grd zd.grd
%
```

D1 = lower bound on fault depth (negative)
D2 = upper bound on fault depth (negative)
Zobs = observation plane depth (negative)
elements.xyF = position of fault elements and force magnitudes (x1 x2 y1 y2 F1 F2 F3)
xd.grd = name of x-displacement grd file (or txx-stress grd file)
yd.grd = name of y-displacement grd file (or tyy-stress grd file)
zd.grd = name of z-displacement grd file (or txy-stress grd file)

```
c
      program fftfault
c
c  program to calculate the surface displacement or stress
c  due to planar vertical fault element(s) between depth D1 and D2.
c  the size of the area is changed by modifying ni and nj in the
c  parameter statement below.
c
c*****************   main program   ***********************
      implicit complex (c)
      real kx,ky
      real*8 rln0,rlt0,ddx,ddy,rland,rdum
      parameter(ni=2048,nj=2048,nwork=32768,nj2=nj/2+1,ni2=ni/2+1)
      character*80 felement,fxdisp,fydisp,fzdisp
      character*80 cd1,cd2,cz,cstr
c
      common/elastic/rlam,rmu,rc,rd,alph,pi
c
      dimension fx(nj,ni),fy(nj,ni),fz(nj,ni)
      dimension u(nj,ni),v(nj,ni),w(nj,ni)
      complex fkx(nj2,ni),fky(nj2,ni),fkz(nj2,ni)
      complex uk(nj2,ni),vk(nj2,ni),wk(nj2,ni),Tk
      dimension n(2)
      complex work(nwork)
      equivalence (fx(1,1),fkx(1,1))
      equivalence (fy(1,1),fky(1,1))
      equivalence (fz(1,1),fkz(1,1))
      equivalence (u(1,1),uk(1,1))
      equivalence (v(1,1),vk(1,1))
      equivalence (w(1,1),wk(1,1))
c
      pi=acos(-1.)
c
```

```
c  zero the arrays fxdisp,fydisp,fzdisp
c
      do 30 i=1,ni
      do 30 j=1,nj
       fx(j,i)=0.
       fy(j,i)=0.
       fz(j,i)=0.
  30  continue
c  set the dimensions for fourt
c
      n(1)=nj
      n(2)=ni

30  continue
c
c   get values from command line
c
      narg = iargc()
      if(narg.lt.8) then
        write(*,'(a)')'  '
        write(*,'(a)')
    &  'Usage: fftfault D1 D2 Zobs ele.dat istr xd.grd yd.grd zd.grd'
        write(*,'(a)')
        write(*,'(a)')
     &  '       D1, D2  - depth to bottom and top of fault (neg) '
        write(*,'(a)')
     &  '       Zobs    - depth of observation plane '
        write(*,'(a)')
     &  '       ele.dat - file of: x1,x2,y1,y2,F1,F2,F3'
        write(*,'(a)')
     &  '       istress - (0)-disp. U,V,W or (1)-stress Txx,Tyy,Txy'
        write(*,'(a)')
     &  '       x,y,z.grd - output files of disp. or stress '
        write(*,'(a)')'  '
        stop
      else
        call getarg(1,cd1)
        call getarg(2,cd2)
        call getarg(3,cz)
        call getarg(5,cstr)
        read(cd1,*)d1
        read(cd2,*)d2
        read(cz,*)zobs
        read(cstr,*)istr
        if(d2.gt.0..or.d1.ge.d2.or.zobs.lt.d2.or.zobs.gt.0) then
         write(*,'(a)')' depths should be negative and d1 < d2 < zobs'
         stop
        endif
        call getarg(4,felement)
        call getarg(6,fxdisp)
        nc=index(fxdisp,' ')
        fxdisp(nc:nc)='\0'
        call getarg(7,fydisp)
        nc=index(fydisp,' ')
        fydisp(nc:nc)='\0'
        call getarg(8,fzdisp)
        nc=index(fzdisp,' ')
        fzdisp(nc:nc)='\0'
      endif
```

```
c
c  open the input file and load the force arrays
c
      open(unit=5,file=felement,status='old')
      read(5,*)cd1
c
c  read the source constants
c
      read(5,*) dx,sig,rlam,rmu,ideep
      width=dx*nj
      height=dx*ni
      alph=(rlam+rmu)/(rlam+2*rmu)
      rc=alph/(4.*rmu)
      rd=(rlam+3*rmu)/(rlam+rmu)
      sig=sig/dx
      if(sig.lt.1)sig=1
c
c  read the elements
c
   50 read(5,*,end=999)x1,x2,y1,y2,F1,F2,F3
      x1=x1/dx
      x2=x2/dx
c
c  invert the y-positions with i-index
c
      y1=ni-y1/dx
      y2=ni-y2/dx

figure out the part of the array to add the new forces
c
      ipd=3*sig+2
      jx0=min(x1,x2)-ipd
      jxf=max(x1,x2)+.5+ipd
      iy0=min(y1,y2)-ipd
      iyf=max(y1,y2)+.5+ipd
      do 100 iy=iy0,iyf
      do 200 jx=jx0,jxf
        if(jx.le.0.or.jx.gt.nj) go to 200
        if(iy.le.0.or.iy.gt.ni) go to 200
        x=jx
        y=iy
        call element(x,y,x1,y1,x2,y2,dx,sig,F1,F2,F3,fxx,fyy,fzz)
        fx(jx,iy)=fx(jx,iy)+fxx
        fy(jx,iy)=fy(jx,iy)+fyy
        fz(jx,iy)=fz(jx,iy)+fzz
c
c
c remove the x-forces if the fault goes deep
c
        if(ideep.eq.1)fx(jx,iy)=0.
 200  continue
 100  continue
      go to 50
 999  close(unit=5)
c
c  if this is a deep source then mirror the source in the x-direction
c  this will make the zero-slope but finite displacement BC's match
c  one could also use a cosine transform
```

```
c
      if(ideep.eq.1) then
        do 150 iy=1,ni
        do 150 jx=1,nj2
        fx(nj+1-jx,iy)=-fx(jx,iy)
        fy(nj+1-jx,iy)= fy(jx,iy)
        fz(nj+1-jx,iy)= fz(jx,iy)
 150    continue
      endif
c
c  take the fourier transform of the forces
      call fourt(fx,n,2,-1,0,work,nwork)
      call fourt(fy,n,2,-1,0,work,nwork)
      call fourt(fz,n,2,-1,0,work,nwork)
c
c  construct the fault solution
c
      do 255 i=1,ni
      ky=-(i-1)/height
      if(i.ge.ni2) ky= (ni-i+1)/height
      do 255 j=1,nj2
      kx=(j-1)/width
      call fault(kx,ky,d1,d2,zobs,cux,cuy,cuz,
     +                            cvx,cvy,cvz,
     +                            cwx,cwy,cwz,
     +                            c3x,c3y,c3z)
      uk(j,i)=(fkx(j,i)*cux+fky(j,i)*cuy+fkz(j,i)*cuz)/(ni*nj)
      vk(j,i)=(fkx(j,i)*cvx+fky(j,i)*cvy+fkz(j,i)*cvz)/(ni*nj)
      wk(j,i)=(fkx(j,i)*cwx+fky(j,i)*cwy+fkz(j,i)*cwz)/(ni*nj)
      Tk     =(fkx(j,i)*c3x+fky(j,i)*c3y+fkz(j,i)*c3z)/(ni*nj)

c  do the boussinesq problem
c
      call boussinesq(kx,ky,zobs,cub,cvb,cwb)
c
c  now either output displacement or stress
c
      uk(j,i)=rmu*(uk(j,i)+Tk*cub)
      vk(j,i)=rmu*(vk(j,i)+Tk*cvb)
      wk(j,i)=rmu*(wk(j,i)+Tk*cwb)
c
c  now compute the stress if requested.
c
      if(istr.eq.1) then
        cux=cmplx(0.,2.*pi*kx)*uk(j,i)
        cvy=cmplx(0.,2.*pi*ky)*vk(j,i)
        cuy=cmplx(0.,2.*pi*ky)*uk(j,i)
        cvx=cmplx(0.,2.*pi*kx)*vk(j,i)
        cwz=rlam*(cux+cvy)/(rlam+2.*rmu)
c
c  assume units are km so divide by 1000
c  units originally in m, take derivative wrt km, get m/km
c  divide by 1000 in order to get m/m
c  need to divide by another 1000 because answer in mm
c
        uk(j,i)=.001*.001*((rlam+2*rmu)*cux+rlam*(cvy+cwz))
        vk(j,i)=.001*.001*((rlam+2*rmu)*cvy+rlam*(cux+cwz))
        wk(j,i)=.001*.001*(rmu*(cuy+cvx))
```

```
       endif
 255  continue
c
c  do the inverse fft's

 256
c
      call fourt(u,n,2,1,-1,work,nwork)
      call fourt(v,n,2,1,-1,work,nwork)
      call fourt(w,n,2,1,-1,work,nwork)
c
c  write 3 grd files some parameters must be real*8
c
      rlt0=0.
      ddx=dx
      ddy=dx
      rland=9998.
      rdum=9999.
      if(istr.eq.1) then
       rland=9998.d0*rmu
       rdum=9999.d0*rmu
      endif
      call writegrd(u,nj,ni,rln0,rlt0,ddx,ddy,
     +              rland,rdum,fxdisp,fxdisp)
      call writegrd(v,nj,ni,rln0,rlt0,ddx,ddy,
     +              rland,rdum,fydisp,fydisp)
      call writegrd(w,nj,ni,rln0,rlt0,ddx,ddy,
     +              rland,rdum,fzdisp,fzdisp)

            stop
      end
      subroutine element(x,y,x1,y1,x2,y2,dxx,sig,F1,F2,F3,Fx,Fy,Fz)
c
      common/elastic/rlam,rmu,rc,rd,alph,pi
c       create the force vector for this element
c
c       input:
c
c       x, y    - computation point      (km)
c       x1,y1   - start position         (km)
c       x2,y2   - end position           (km)
c       dx      - grid spacing           (km)
c       sig     - source width in pixels (try 1)
c       F1      - strike-slip
c       F2      - dip-slip
c       F3      - opening mode
c
c       output:
c
c       fxx     - x-component of force couple
c       fyy     - y-component of force couple
c       fzz     - z-component of force couple
c
c  set the x and y taper lengths
c
      sigx=sig
      sigy=2
c
c  compute the length and orientation of the fault
c
      Dx=(x2-x1)
      Dy=(y2-y1)
      L=sqrt(Dx*Dx+Dy*Dy)
      theta=atan2(Dy,Dx)
      st=sin(theta)
      ct=cos(theta)
c
c  rotate the vector into the model space
c
      x0=x-x1
      y0=y-y1
      xp=x0*ct+y0*st
      yp=-x0*st+y0*ct
c
c  compute the h and dh/dx function
c
      h=exp(-0.5*((yp/sigx)**2))/(sqrt(2*pi)*sigx)/(dxx*dxx)
      dh=-yp*exp(-0.5*((yp/sigx)**2))/(sqrt(2*pi)*sigx**3)/(dxx*dxx)
c
c   compute the g function
c
      if(xp .gt. -sigy .and. xp .lt. sigy) then
         g=0.5*(1-cos(pi*(xp+sigy)/(2*sigy)))
      else if(xp .ge. sigy .and. xp .le. L-sigy) then
         g=1.
      else if(xp .gt. L-sigy .and. xp .lt. L+sigy) then
         g=0.5*(1-cos(pi*(L+sigy-xp)/(2*sigy)))
      else
         g=0.
      endif
```

```
c extract components
c
      Fx=g*(dh*(F1*ct+F3*st))
      Fy=g*(dh*(F1*st-F3*ct))
      Fz=g*(dh*F2)
      return
      end
```

```
      subroutine fault(kx,ky,d1,d2,z,cux,cuy,cuz,
     +                               cvx,cvy,cvz,
     +                               cwx,cwy,cwz,
     +                               c3x,c3y,c3z)
c
c  input
c    kx  - x wavenumber        (1/km)
c    ky  - y wavenumber        (1/km)
c    d1  - bottom of fault d1 < d2 < 0. (km)
c    d2  - top of fault        (km)
c    z   - observation plane   (km)
c
c  output
c    cux,cuy,cuz  - elements of force/displacement matrix
c    cvx,cvy,cvz               (km**2/Pa)
c    cwx,cwy,cwz
c    c3x,c3y,c3z  - non-zero vertical stress at surface z=0
c                              (km)
c
      implicit complex (c)
      implicit real*4 (k)
c
c  routine to compute the greens function for a fault element
c
      common/elastic/rlam,rmu,rc,rd,alph,pi
c
      if (kx.eq.0.and.ky.eq.0.) then
        cux=cmplx(0.,0.)
        cuy=cmplx(0.,0.)
        cuz=cmplx(0.,0.)
        cvx=cmplx(0.,0.)
        cvy=cmplx(0.,0.)
        cvz=cmplx(0.,0.)
        cwx=cmplx(0.,0.)
        cwy=cmplx(0.,0.)
        cwz=cmplx(0.,0.)
        cwz=cmplx(0.,0.)
        c3y=cmplx(0.,0.)
        c3z=cmplx(0.,0.)
        c3z=cmplx(0.,0.)
      else
        kh2=kx*kx+ky*ky
        kh=sqrt(kh2)
        B=2*pi*kh
        B2=B*B
        Bs1=B*(z-d1)
        Bs2=B*(z-d2)
        Bi1=B*(z+d1)
        Bi2=B*(z+d2)
```

```
if(Bs1.gt.50.) then
          es1=0.
        else
          es1=exp(-Bs1)
        endif
        if(Bs2.gt.50.) then
          es2=0.
        else
          es2=exp(-Bs2)
        endif
        if(Bi1.lt.-50.) then
          ei1=0.
        else
          ei1=exp(+Bi1)
        endif
        if(Bi2.lt.-50.) then
          ei2=0.
```

```
      subroutine boussinesq(kx,ky,z,cub,cvb,cwb)
c
c  input
c    kx  - x wavenumber        (1/km)
c    ky  - y wavenumber        (1/km)
c    z   - observation plane   (km)
c
c  output
c    cub - x-displacement      (km/Pa)
c    cvb - y-displacement
c    cwb - z-displacement
c
      implicit complex (c)
      implicit real*4 (k)
c
c  routine to compute the greens function for a fault element
c
      common/elastic/rlam,rmu,rc,rd,alph,pi
c
      if (kx.eq.0.and.ky.eq.0.) then
        cub=cmplx(0.,0.)
        cvb=cmplx(0.,0.)
        cwb=cmplx(0.,0.)
      else
        kh2=kx*kx+ky*ky
        kh=sqrt(kh2)
       B=2*pi*kh
        B2=B*B
        arg=B*z
        if(arg.lt.-50.) then
          eb=0.
        else
          eb=exp(arg)
        endif
        arg1=(1.-1./alph-B*z)/(2.*rmu)
       arg2=(   1./alph-B*z)/(2.*rmu)
        cub=cmplx(0.,-2.*pi*kx*arg1*eb/B2)
        cvb=cmplx(0.,-2.*pi*ky*arg1*eb/B2)
        cwb=cmplx(-arg2*eb/B)
      endif
      return
      end
```